The strength of rigid/plastic composites: a comparison of piecewise-linear and power-law approximations

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Abstract

In theoretical models of material response, rigid/ideally plastic behavior is often viewed as as special limiting case of power-law materials. In this work, we examine another constitutive relation which also has rigid/ideally plastic behavior as a limiting case. In particular, our analysis deals with the overall properties of a class of composites where the stress/strain(rate) relation is piecewise linear in each constituent material ("bilinear" response).

When comparing our work to previous analysis of power-law materials in the rigid/ideally plastic limit, the results can be strikingly different. For example, adding a small amount of stronger material to a weaker one can actually result in a composite with a lower yield stress than the original (weaker) material. We will discuss the discrepancies between the two limits and the circumstances in which the limits agree.

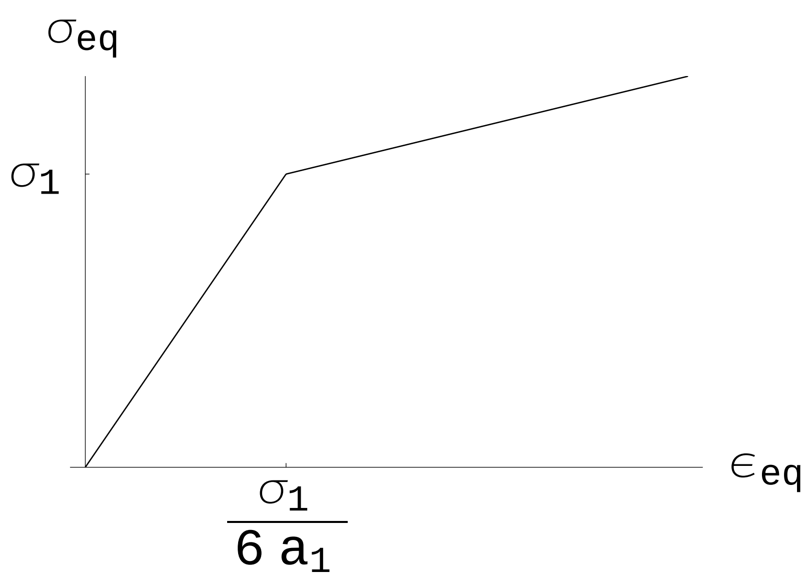
The Materials — Bilinear

- Incompressible, isotropic:
  potential depends only on $\epsilon_{eq}^2 = \frac{2}{3}\epsilon_{ij}\epsilon_{ij}$
- Two materials with parameters (a_1, b_1, σ_1) and (a_2, b_2, σ_2)

$$\phi_1(\epsilon) = \begin{cases} 3a_1(\epsilon_{eq})^2 & \text{if } \epsilon_{eq} \leq \frac{\sigma_1}{6a_1} ; \\ 3b_1(\epsilon_{eq})^2 + (1 - \frac{a_1}{b_1})\sigma_1\, \epsilon_{eq} - \frac{(a_1-b_1)\sigma_1^2}{12a_1^2} & \text{if } \epsilon_{eq} \geq \frac{\sigma_1}{6a_1} , \end{cases} \tag{1}$$

or

$$\sigma_{ij} = \begin{cases} 4a_1(\epsilon_{ij}) & \text{if } \epsilon_{eq} \leq \frac{\sigma_1}{6a_1} ; \\ 4b_1\epsilon_{ij} + \frac{2(a_1-b_1)\sigma}{3a_1\epsilon_{eq}}\, \epsilon_{ij} & \text{if } \epsilon_{eq} \geq \frac{\sigma_1}{6a_1} . \end{cases} \tag{2}$$



$\sigma_{eq}^2 = \frac{3}{2}\sigma_{ij}\sigma_{ij}$

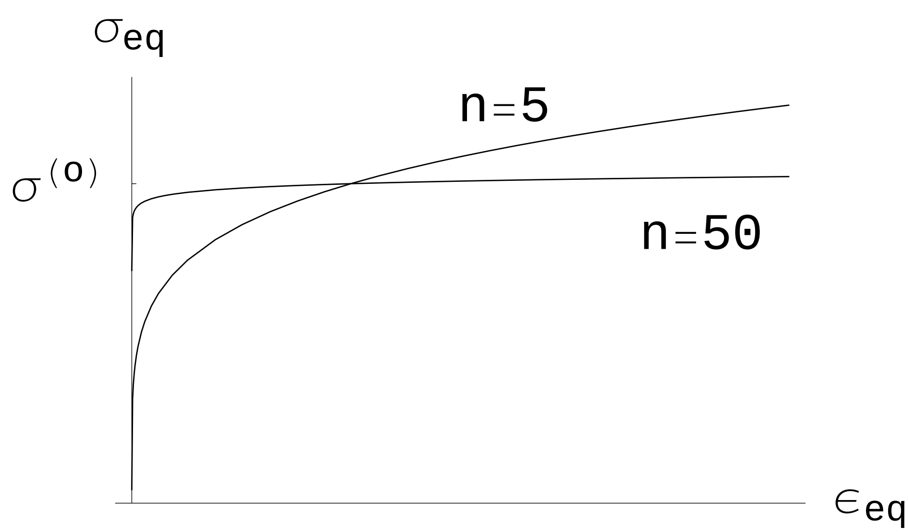
Yield limit: $a_1 \to \infty$, $b_1 \to 0$

The Materials — Power-Law

- Two incompressible, isotropic power-law materials
- parameters $(\sigma_1^{(0)}, m = 1/n)$ and $(\sigma_2^{(0)}, m = 1/n)$
  ** same exponent "m" in both materials **

$$\sigma_{ij} = \frac{2}{3}\sigma_0 \epsilon_{eq}^{m-1} \epsilon_{ij} \quad (3)$$

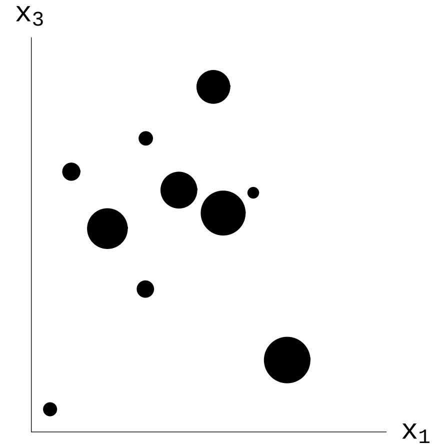
$$\phi(\boldsymbol{\epsilon}) = \frac{\sigma_0}{m+1} \boldsymbol{\epsilon}_{eq}^{m+1} \qquad \text{if } Tr(\boldsymbol{\epsilon}) = 0, \quad +\infty \text{ otherwise} \quad (4)$$



Yield limit: $n \to \infty$ $(m \to 0)$

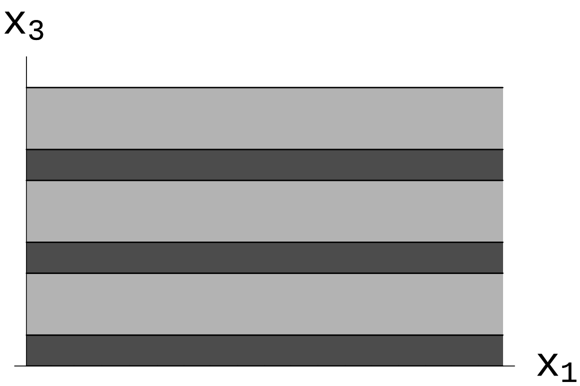
The Geometries

Spherical Inclusions



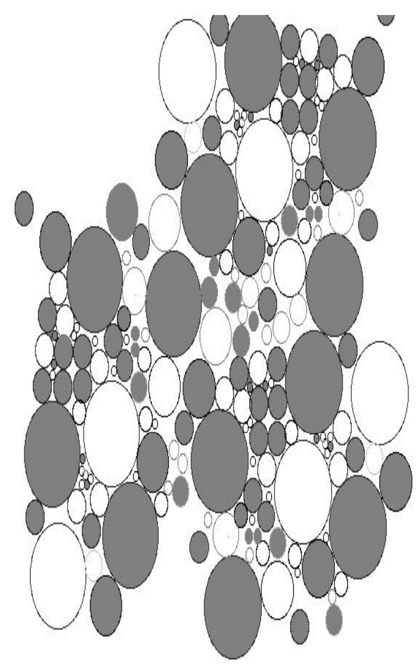
- spherical inclusions
- randomly distributed
- inclusions stiffer than matrix
- overall isotropic

Laminate



- Layers perpendicular to $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- overall transversely isotropic

Ellipsoids



- aligned ellipsoids with circular cross-section perpendicular to $\vec{n}$
- Space-filling ellipsoids of both phases (no contiguous matrix material)
- Materials statistically interchangeable
- Aspect ratio "x": x_3/x_1 axis ratio
  Disks: $x = 0$
  Spheres: $x = 1$
  Needles: $x = \infty$
- overall transversely isotropic

The Variational Method

In a mixture, the average stress $<\boldsymbol{\sigma}>$ depends on the average applied strain $\boldsymbol{\epsilon}_0 = <\boldsymbol{\epsilon}>$ through $\Phi(\boldsymbol{\epsilon}_0)$, the overall potential of the bilinear mixture:

$$<\boldsymbol{\sigma}> = \Phi'(\boldsymbol{\epsilon}_0).$$

We consider two possible boundary conditions,

$$\boldsymbol{\epsilon}_0 = \alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \boldsymbol{\epsilon}_0 = \alpha \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

"in-plane strain" and "anti-plane strain."

We use the Ponte Castañeda variational inequality [Pon91]:

$$\Phi(\boldsymbol{\epsilon}_0) \ \leq \ \Phi_0(\boldsymbol{\epsilon}_0) + < \sup_A \left[\phi(x,A) - \phi_0(x,A)\right] > \tag{5}$$

with an inhomogeneous linear comparison material:

$$\phi_0(x,A) = \begin{cases} \frac{3}{2}\mu_1 (A_{eq})^2 & \text{in material 1} \\ \frac{3}{2}\mu_2 (A_{eq})^2 & \text{in material 2} \end{cases}$$

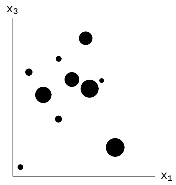
to compute an UPPER BOUND on the overall potential $\Phi(\boldsymbol{\epsilon}_0)$.

For bilinear materials:

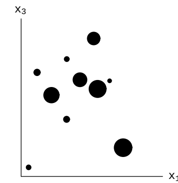
For small applied strain ($\alpha << 1$), the upper bound on the potential is quadratic and the stress-strain relation is linear. We increase the applied strain, α, until this ceases to be the case.

Define σ^*:
the *first point of nonlinearity* in the upper bound.

In the yield limit ($a_1, a_2 \to \infty$), σ^* is an upper bound on the yield stress.



Results — Spherical Inclusions



Using the linear bound of Hashin [Has83] for $\Phi_0(\boldsymbol{\epsilon}_0) = \boldsymbol{\epsilon}_0 \cdot C^* \boldsymbol{\epsilon}_0$:

$$C^* \leq \frac{\mu_2\left((5f_1 + 2f_2)(\mu_1 - \mu_2) + 5\mu_2\right)}{2f_2(\mu_1 - \mu_2) + 5\mu_2}, \qquad \mu_1 < \mu_2, \tag{6}$$

we obtain

$$\sigma^* = (2 + 3f_1 + 3\frac{a_2}{a_1}f_2) \cdot \min\left\{\frac{\sigma_1}{5}, \frac{\sigma_2}{\sqrt{(2 + 3\frac{a_2}{a_1})^2 + 6(1 - \frac{a_2}{a_1})^2 f_1}}\right\} \tag{7}$$

$$a_1 < a_2.$$

- If $\sigma_2 > \sigma_1$, σ^* is a decreasing function of $\frac{a_2}{a_1}$ when

  $$\frac{a_2}{a_1} > \frac{6(f_1 - 1)\sigma_1 + 5\sqrt{3}\sqrt{2f_1(\sigma_2^2 - \sigma_1^2) + 3\sigma_2^2}}{3(2f_1 + 3)\sigma_1}$$

  and increasing otherwise.

- If $a_2 < a_1$, σ^* is an increasing function of f_1, even when $\sigma_2 > \sigma_1$ (!!).

Results — Laminate

Using the potential for a laminate of linear materials (see [Suq93] or [Mil02]) we obtain

$$\sigma^* \;=\; (f_1 + f_2\frac{a_2}{a_1})\;\min\left\{\sigma_1, \frac{a_1}{a_2}\sigma_2\right\} \qquad \text{for } (\epsilon_0)_{12} \neq 0, \tag{8}$$

and

$$\sigma^* \;=\; \min\{\sigma_1, \sigma_2\} \qquad \text{for } (\epsilon_0)_{13} \neq 0. \tag{9}$$

For $\boldsymbol{\epsilon}_0 = \alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, notice:

- Classical yield strength is the arithmetic mean:
  (as is the limit of power-law material [Suq93])

$$\sigma^*_{Suquet} = f_1\sigma_1 + f_2\sigma_2 \tag{10}$$

- The bilinear bound is tighter

$$\sigma^* \leq \sigma^*_{Suquet} \tag{11}$$

  with equality only when $\frac{\sigma 2}{\sigma 1} = \frac{a_2}{a_1}$.

- If $\frac{a_2}{a_1} << \frac{\sigma_2}{\sigma_1}$, $\sigma^* \approx f_1\sigma_1$
- If $\frac{a_2}{a_1} >> \frac{\sigma_2}{\sigma_1}$, $\sigma^* \approx f_2\sigma_2$
- Consider $\sigma_1 < \sigma_2$. If $\frac{a_2}{a_1} < 1$ and $\frac{a_2}{a_1} < \frac{\sigma_2}{\sigma_1}$, we have

$$\sigma^* = (f_1 + f_2\frac{a_2}{a_1})\sigma_1 \tag{12}$$

  so that

$$\sigma^* < \sigma_1. \tag{13}$$

- With the above conditions, σ^* is an **in**creasing function of f_1.

Results — Ellipsoids

Using the results of Eshelby [Esh57] to evaluate the linear bound of Willis [Wil81] for $\Phi_0(\boldsymbol{\epsilon}_0) = \boldsymbol{\epsilon}_0 \cdot C^* \boldsymbol{\epsilon}_0$:

$$C^* \;\leq\; \mu_2 \left(I - f_1 \left(\frac{\mu_2}{\mu_2 - \mu_1} I - f_2 S^{Esh} \right)^{-1} \right) \tag{14}$$

we obtain

$$\begin{aligned} \sigma^* \;=\; & \left((f_1 + f_2 s) + \frac{a_2}{a_1} f_2 (1-s) \right) \cdot \\ & \min \left\{ \sigma_1, \frac{\sigma_2}{\sqrt{s + (1-s)((\frac{a_2}{a_1})^2 - (1 - \frac{a_2}{a_1})^2 f_2 s))}} \right\} \end{aligned} \tag{15}$$

$s = 2S^{Esh}_{ijij}$ (twice the $\{ijij\}$ component of Eshelby's tensor)

Disks, $(\epsilon_0)_{12} \neq 0$: $s = 0$, Spheres: $s = \frac{2}{5}$
Disks, $(\epsilon_0)_{13} \neq 0$: $s = 1$, Needles: $s = \frac{1}{2}$

- Identical to laminate result as $x \to 0$ (disks)
- Identical to spherical inclusions result as $x \to 1$
- Always tighter than classical yield result [Ols98], with equality only when

$$\frac{a_2}{a_1} = \frac{-f_2 s}{1 - f_2 s} + \frac{\sqrt{(1-s)\left((1 - f_2 s)\sigma_2^2 - f_1 s \sigma_1^2\right)}}{(1-s)(1-f_2 s)\sigma_1} \tag{16}$$

References

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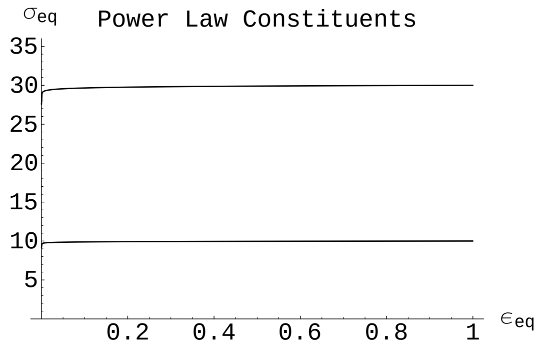
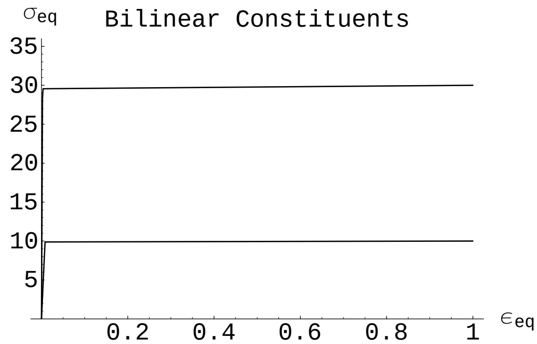


Limiting case: yield stress
(laminate with f1=0.2, 12−component non−zero): **14**

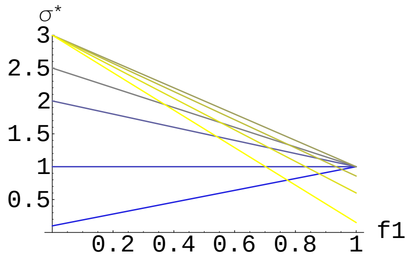


Limiting case: yield stress
(laminate with f1=0.2, 12−component non−zero): **8.4**

σ^* vs. f_1
$\epsilon_{12} \neq 0$, $\sigma_1 = 1$, $\sigma_2 = 3$

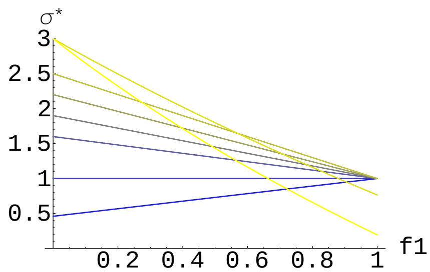
Laminate

a2/a1 values: {0.1, 1., 2, 2.5, 3, 3.5, 5, 20}



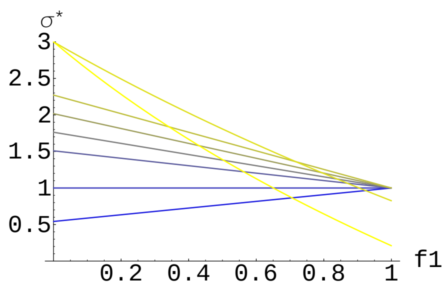
Spheres

a2/a1 values: {0.1, 1., 2, 2.5, 3, 3.5, 5, 20}

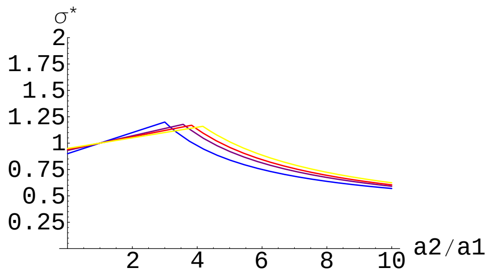


Needles

a2/a1 values: {0.1, 1., 2, 2.5, 3, 3.5, 5, 20}

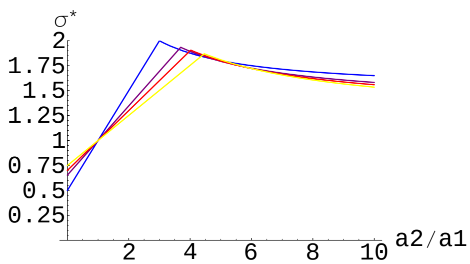


σ^* vs. a2 / a1
$\epsilon_{12} \neq 0$, $\sigma_1 = 1$, $\sigma_2 = 3$

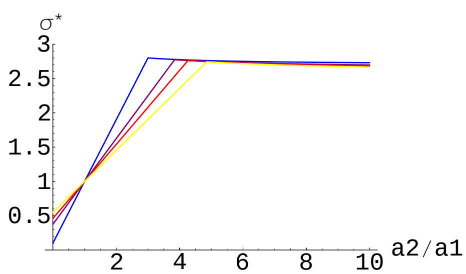


f_1 = 0.9

for laminate, oblate ellipsoids, spheres, needles (left->right)



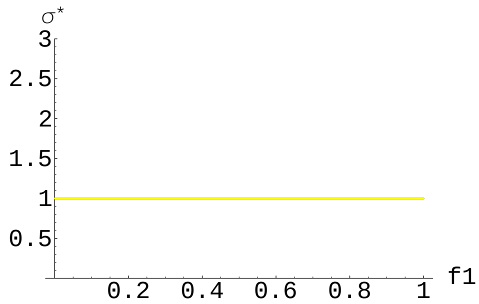
f_1 = 0.5



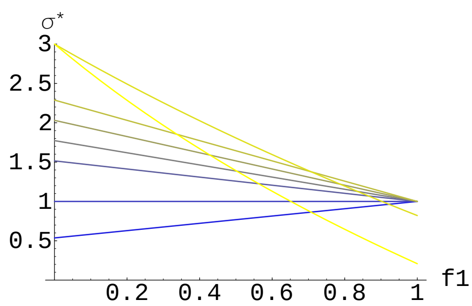
f_1 = 0.1

σ^* vs. f_1
$\epsilon_{13} \neq 0$, $\sigma_1 = 1$, $\sigma_2 = 3$

Laminate (any a_i values)



Oblate ellipsoids



a2 / a1 values : {0.1, 1., 2, 2.5, 3, 3.5, 5, 20}

Spheres, Needles