(k, j)-Colored Partitions and The Han/Nekrasov-Okounkov Hooklength Formula

Emily Anible[1] William J. Keith[1]

[1]Michigan Technological University

Spring 2018

Partitions — Definition

A partition of n is a weakly decreasing sequence of positive integers who sum to n, given by $\lambda = (\lambda_1, \ldots, \lambda_j)$.
The number of partitions of n is given by the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q)_\infty} = \prod_{i=1}^{\infty} \frac{1}{1-q^i}, \quad p(0) = 1,$$

where the coefficient on the q^k term counts the number of ways to write $k = 1a_1 + 2a_2 + 3a_3 + \ldots$, where the coefficient i appears a_i times.

•$(q)_\infty = (q;q)_\infty = \prod_{i=1}^{\infty}(1-q^i)$ is the Q-Pochhammer symbol.

Partitions — Definition

•$\lambda \vdash n$ denotes that λ partitions n.
•We will make frequent use of the notation $\lambda = 1^{\nu_1}2^{\nu_2}3^{\nu_3}\ldots$, which denotes a partitions with ν_1 parts of size 1, ν_2 parts of size 2, and so on. So, the partitions of 3 can be given by the following:

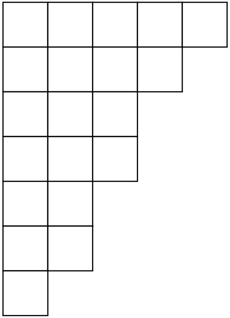
$$(3), (2, 1), (1, 1, 1)$$

or

$$3^1, 1^1 2^1, 1^3$$

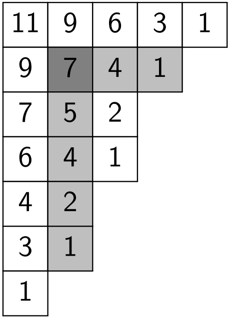
Partitions — Ferrers Diagram

The *Ferrers Diagram* of a partition is a stack of unit-sized squares justified to the origin in the fourth quadrant.
For example, the Ferrers diagram of a partition of 20,
$\lambda = (5, 4, 3, 3, 2, 2, 1)$, is



Partitions — Hooklengths

The hook length h_{ij} of the square with lower right corner at $(-i, -j)$ in the plane is the count of the number of squares both to its right and directly below it in the Ferrers diagram, $+1$ for the square of interest itself. For example, the hook length of the $(-2, -2)$ square is 7:



Partitions — Hooklengths

The conjugate of a partition λ, denoted by λ', is the partition of λ reflected across the diagonal (in our plane description, across the line $y = -x$).
A partition fixed under conjugation is a self-conjugate partition.
The conjugate of $\lambda = (5, 4, 3, 3, 2, 2, 1)$ is $\lambda' = (7, 6, 4, 2, 1)$.

$\lambda =$

11	9	6	3	1
9	7	4	1	
7	5	2		
6	4	1		
4	2			
3	1			
1				

$\lambda' =$

11	9	7	6	4	3	1
9	7	5	4	2	1	
6	4	2	1			
3	1					
1						

Other Notation

Some other notation that will be used throughout the presentation:

$$\nu_k(n) = \sum_{\lambda \vdash n} \nu_k : \text{ number of parts of size } k \text{ in the partitions of } n.$$

$$\gamma_k(n) = \sum_{\lambda \vdash n} \gamma_k : \text{ number of part sizes with frequency } k$$

in the partitions of n.

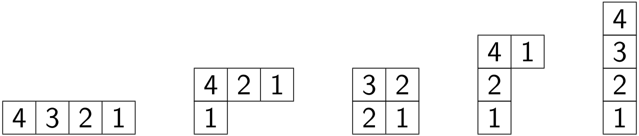
$$\gamma_{\geq k}(n) = \sum_{\lambda \vdash n} \gamma_{\geq k} : \text{ number of part sizes with frequency of at least } k$$

in the partitions of n.

We will occasionally use $\ell_0(\lambda)$ to refer to the number of distinct parts in a partition. This is equal to $\gamma_{\geq 1}$.

Other Notation

For example, the partitions of 4 are:



$\nu(4) = (7, 3, 1, 1)$, where $\nu_1(4) = 7$: There are seven parts of size 1 in the partitions of 4.
$\gamma(4) = (4, 2, 0, 1)$, where $\gamma_2(4) = 2$: There are two part sizes that appear exactly twice in the partitions of 4.

It is known that $\nu_k(n) = \sum_{i=k}^{n} \gamma_i(n) = \gamma_{\geq k}(n)$, that is, the number of parts of size k over all partitions of n is equal to the number of frequencies greater than or equal to k in the partitions of n. (Bacher & Manivel [3])

k-colored Partitions — Definition

Partitions in which each part may be assigned one of k available colors, with the orders of the colors not mattering. The 2-colored partitions of 3 are:

$$
\begin{aligned}
&3_2, 3_1,\\
&2_2 + 1_2, 2_2 + 1_1, 2_1 + 1_2, 2_1 + 1_1,\\
&1_2 + 1_2 + 1_2, 1_2 + 1_2 + 1_1, 1_2 + 1_1 + 1_1, 1_1 + 1_1 + 1_1.
\end{aligned}
$$

k-colored Partitions — Generating Function

The generating function of the number of k-colored partitions of n, $c_k(n)$, is formed by simply raising the generating function for a partition to the kth power:

$$C_k(q) := \sum_{n=1}^{\infty} c_k(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k}.$$

Overpartitions

In the same vein as k-colored partitions are an object known as overpartitions — partitions in which we either mark or don't mark the last part of a given size. The number of overpartitions of n, given by $\bar{p}(n)$, is:

$$\bar{P}(q) := \sum_{n=0}^{\infty} \bar{p}(n) q^n = \prod_{k=1}^{\infty} \frac{1+q^k}{1-q^k} = \prod_{k=1}^{\infty} \frac{1-q^{2k}}{(1-q^k)^2}.$$

Plenty of research has been done on overpartitions, and Keith has related them to the k-colored partitions by extending the latter to the (k,j)-colored partitions.

(k, j)-colored Partitions

(k, j)-colored partitions extend k-colored partitions by limiting us to using at most j of the k available colors for a given size of part. The generating function for $c_{k,j}(n)$, the number of (k, j)-colored partitions, is given by:

$$
\begin{aligned}
C_{k,j}(q) &:= \sum_{n=0}^{\infty} c_{k,j}(n) q^n \\
&= \prod_{n=1}^{\infty} \left(1 + \frac{\binom{k}{1} q^n}{1 - q^n} + \frac{\binom{k}{2} q^{2n}}{(1 - q^n)^2} + \cdots + \frac{\binom{k}{j} q^{jn}}{(1 - q^n)^j} \right) \\
&= \frac{1}{(q)_\infty{}^j} \prod_{n=1}^{\infty} \left(\sum_{i=0}^{j} \binom{k}{i} (1 - q^n)^{j-i} q^{in} \right).
\end{aligned}
$$

(k, j)-colored Partitions

$$c_{k,j}(n)q^n = \frac{1}{{(q)_\infty}^j} \prod_{n=1}^{\infty} \left(\sum_{i=0}^{j} \binom{k}{i} (1-q^n)^{j-i} q^{in} \right).$$

For each i, we get a choice of i of the k available colors, then count the parts of size n with the i chosen colors for the partition.

(k, j)-colored Partitions

(k, j)-colored partitions are able to generalize other partitions that label parts.

ex: Overpartitions are the $(2, 1)$-colored partitions, where we only allow ourselves to mark any one part of each size, regardless of position in the partition.

HNO Hooklength Formula

Much research has been done on k-colored partitions, generalizing them, and finding relations to other types of partitions. One such interesting relation is that of (k, j)-colored partitions and the Han/Nekrasov-Okounkov Hooklength formula.

HNO Hooklength Formula — Definition

HNO expands the product below, giving coefficients on q^n as polynomials in b, a complex indeterminate.

$$\sum_{n=0}^{\infty} p_n(b) q^n := \prod_{n=1}^{\infty} (1 - q^n)^{b-1} = \sum_{n=0}^{\infty} q^n \sum_{\lambda \vdash n} \prod_{h_{i,j} \in \lambda} (1 - \frac{b}{h_{i,j}^2}).$$

This was found by Guo-Niu Han, as well as Nikita Nekrasov and Andrei Okounkov.

HNO and (k, j)

Notice that if in $C_{k,j}(q)$ we let $k = 1 - b$ and let j increase without bound (*i.e.* go to infinity), we get the following for $C_{1-b,\infty}(q)$:

$$C_{1-b,\infty}(q) = \prod_{n=1}^{\infty} \sum_{i=0}^{\infty} \binom{k}{i} \frac{q^{in}}{(1-q^n)^i} = \prod_{n=1}^{\infty} \left(1 + \frac{q^n}{1-q^n}\right)^{1-b}$$

$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n}\right)^{1-b} = \prod_{n=1}^{\infty} (1-q^n)^{b-1}$$

This is exactly the Han-Nekrasov/Okounkov Hooklength Formula.

Truncation

These two objects are equivalent when left unrestricted, what happens when we truncate both of them?

— For $C_{1-b,j}$, we pick $j \in \mathbb{N}$, restricting the number of colors we can use per part size.

— For HNO, there are several options. The clearest truncation is to limit the hooklengths considered to those equivalent or lesser than j. (Notated HNO_j)

Truncation at $j = 1$

For $j = 1$, the two functions are equal. This makes sense, as the number of hook lengths of 1 is equal to the number of part sizes in a partition.

$$C_{1-b,1}(q) = \prod_{n=1}^{\infty} \sum_{i=0}^{j} \binom{1-b}{i} \frac{q^{in}}{(1-q^n)^i}$$

$$= \prod_{n=1}^{\infty} \left(1 + \frac{(1-b)q^n}{1-q^n}\right) = \prod_{n=1}^{\infty} \frac{1-bq^n}{1-q^n}$$

$$= \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \dots)(1 - bq^n)$$

$$= \prod_{n=1}^{\infty} (1 + (1-b)q^n + (1-b)q^{2n} + \dots)$$

$$HNO_1(q) = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \prod_{\substack{h_{ij} \in \lambda \\ h_{ij} = 1}} (1 - \frac{b}{h_{i,j}^2}) q^n = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} (1-b)^{\ell_0(\lambda)} q^n$$

Truncation at $j = 2$

For $j = 2$, the coefficients on q^n are as follows:

$$\sum_{\lambda \vdash n} \prod_{\substack{h_{ij} \in \lambda \\ h_{ij} \leq 2}} \left(1 - \frac{b}{h_{ij}^2}\right) \quad \textit{and} \quad \sum_{\lambda = 1^{\nu_1} 2^{\nu_2} \cdots \vdash n} (1-b)^{\ell_0(\lambda)} \prod_{\nu_i \geq 2} \left(1 - \frac{b}{2}\right)$$

Keith showed that these are equivalent in their constant and linear terms in b.

For higher j, Keith conjectured from numerical observation that the constant and linear terms in b match in $C_{1-b,j}$ and HNO_j — we have proven this over the semester.

Simplifying $C_{1-b,j}$

In order to solve this conjecture, we observed any patterns in the expansion of $C_{1-b,j}$. One particular thread led us to simplifying the generating function $C_{1-b,j}(q)$ quite a bit!

Theorem 1

Theorem 1.

For $j > 0$:

$$C_{1-b,j}(q) := \sum_{n=0}^{\infty} c_{1-b,j}(n) q^n = \sum_{n=0}^{\infty} q^n \sum_{\substack{\lambda \vdash n \\ \lambda = 1^{\nu_1} 2^{\nu_2} \dots}} \prod_{\nu_i} \binom{\min(j,\nu_i) - b}{\min(j,\nu_i)}$$

Notice in $C_{1-b,j}$, if we encounter a frequency $\nu_i > j$, we simply treat it as being equivalent to frequencies of j.

ex: $\lambda = 1^1 2^3 3^1$ is treated as $\lambda = 1^1 2^2 3^1$ for $j = 2$.

Theorem 1 — Proof

To begin, let's consider the generating function of $C_{1-b,j}(q)$, use the identity $\frac{1}{(1-q^n)^j} = \sum_{p=0}^{\infty} \binom{j+p-1}{j-1} q^{pn}$, and apply Newton's Binomial Theorem to $(1-q^n)^{j-i}$:

$$
\begin{aligned}
& C_{1-b,j}(q) \\
&= \prod_{n=1}^{\infty} \left[\frac{1}{(1-q^n)^j} \sum_{i=0}^{j} \binom{1-b}{i} (1-q^n)^{j-i} q^{in} \right] \\
&= \prod_{n=1}^{\infty} \left[\sum_{p=0}^{\infty} \binom{j+p-1}{j-1} q^{pn} \sum_{i=0}^{j} \binom{1-b}{i} q^{in} \sum_{k=0}^{j-i} (-1)^k \binom{j-i}{k} q^{kn} \right]
\end{aligned}
$$

Theorem 1 — Proof

Let's work with the two inner summations from here, rearranging them for convenience. Notice that k can go off to infinity, as any terms after $j-i$ will be zero. We also swap the indices of the summations, as they will evaluate the same.

$$\sum_{i=0}^{j}\binom{1-b}{i}q^{in}\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}q^{kn}$$

$$=\sum_{k=0}^{j}\sum_{i=0}^{\infty}(-1)^k\binom{1-b}{i}\binom{j-i}{k}q^{(i+k)n}$$

$$=\sum_{i=0}^{j}\sum_{k=0}^{\infty}(-1)^k\binom{1-b}{i}\binom{j-i}{k}q^{(i+k)n}$$

Theorem 1 — Proof

Letting $m = i + k$, $k = m - i$, and then applying the identity $\sum_{i\geq 0}(-1)^i \binom{r}{i}\binom{j-i}{m-i} = \binom{j-r}{m}$

$$= \sum_{m=0}^{j} \sum_{i=0}^{\infty} (-1)^{m-i} \binom{1-b}{i} \binom{j-i}{m-i} q^{(i+m-i)n}$$

$$= \sum_{m=0}^{j} (-1)^m q^{mn} \sum_{i=0}^{\infty} (-1)^i \binom{1-b}{i} \binom{j-i}{m-i}$$

$$= \sum_{m=0}^{j} (-1)^m \binom{b+j-1}{m} q^{mn}$$

Theorem 1 — Proof

Replacing this simplified expression into $C_{1-b,j}$ gives:

$$\prod_{n=1}^{\infty}\left[\sum_{p=0}^{\infty}\binom{j+p-1}{j-1}q^{pn}\sum_{i=0}^{j}\binom{1-b}{i}q^{in}\sum_{k=0}^{j-i}(-1)^k\binom{j-i}{k}q^{kn}\right]$$

$$=\prod_{n=1}^{\infty}\left[\sum_{p=0}^{\infty}\sum_{k=0}^{j}(-1)^k\binom{j+p-1}{j-1}\binom{b+j-1}{k}q^{(p+k)n}\right]$$

Now, let's consider the expression's contribution to $\lambda \vdash N = 1^{\nu_1}2^{\nu_2}3^{\nu_3}\ldots$.

$$\sum_{p=0}^{\infty}\sum_{k=0}^{j}(-1)^k\binom{j+p-1}{j-1}\binom{b+j-1}{k}q^{(p+k)N}$$

Theorem 1 — Proof

$$\frac{\displaystyle\sum_{p=0}^{\infty}\sum_{k=0}^{j}(-1)^k\binom{j+p-1}{j-1}\binom{b+j-1}{k}q^{(p+k)N}}{\begin{aligned}&\prod_{p+k=\nu_i=1}\Big[\binom{j+1-1}{j-1}-\binom{j+0-1}{j-1}\binom{b+j-1}{1}\Big]\\ &*\prod_{p+k=\nu_i=2}\Big[\binom{j+2-1}{j-1}-\binom{j+1-1}{j-1}\binom{b+j-1}{1}\\ &\quad+\binom{j+0-1}{j-1}\binom{b+j-1}{2}\Big]\\ &*\dots*\prod_{p+k=\nu_i\geq j}\Big[\sum_{\ell=0}^{j}(-1)^\ell\binom{j-1+\min(\nu_i,j)-\ell}{j-1}\binom{b+j-1}{\ell}\Big].\end{aligned}}$$

Theorem 1 — Proof

For each $\nu_i \in \lambda$, we have the following, letting $\nu_i^* = \min(\nu_i, j)$ for readability.

$$\sum_{\ell=0}^{\nu_i^*} (-1)^\ell \binom{j-1+\nu_i^*-\ell}{j-1} \binom{b+j-1}{\ell}$$

Using $\binom{n}{k} = \binom{n}{n-k} \ldots$

$$= \sum_{\ell=0}^{\nu_i^*} (-1)^\ell \binom{j-1+\nu_i^*-\ell}{\nu_i^*-\ell} \binom{b+j-1}{\ell}.$$

Theorem 1 — Proof

Again using the identity $\sum_{i\geq 0}(-1)^i\binom{r}{i}\binom{j-i}{m-i} = \binom{j-r}{m}$ gives us our simplified contribution:

$$\begin{aligned}
&\sum_{\ell=0}^{\nu_i^*}(-1)^\ell\binom{(b+j-1)}{\ell}\binom{(j-1+\nu_i^*)-\ell}{(\nu_i^*)-\ell}\\
&=\binom{(j-1+\nu_i^*)-(b+j-1)}{\nu_i^*}\\
&=\binom{\nu_i^*-b}{\nu_i^*}=\binom{\min(j,\nu_i)-b}{\min(j,\nu_i)}.
\end{aligned}$$

Theorem 1 — Proof

Since the contribution to $\lambda \vdash N$ of the inner sums is $\binom{\min(j,\nu_i)-b}{\min(j,\nu_i)}$, we can replace this in $C_{1-b,j}(q)$, greatly simplifying the expression.

$$C_{1-b,j}(q) = \sum_{N=0}^{\infty} q^N \sum_{\substack{\lambda \vdash N \\ \lambda = 1^{\nu_1} 2^{\nu_2} \ldots}} \prod_{\nu_i} \binom{\min(j,\nu_i) - b}{\min(j,\nu_i)}.$$

□

Constant and Linear Equivalence

Now that we have a simplified generating function for $C_{1-b,j}(q)$, it becomes quite easy to prove our original objective:

Theorem 2

Theorem 2.

Let $C_{1-b,j}$ be the count of the $(1-b,j)$-colored partitions of n considering all parts of frequency ν_i greater than j to have frequency j, and let HNO_j be the Han/Nekrasov-Okounkov hooklength formula truncated to only consider hooks less than or equivalent to j. Then the q^i terms of

$$HNO_j(q) = \sum_{n=0}^{\infty} q^n \sum_{\lambda \vdash n} \prod_{\substack{h_{ij} \in \lambda \\ h_{ij} \leq j}} \left(1 - \frac{b}{h_{ij}^2}\right) \qquad \text{and}$$

$$C_{1-b,j}(q) = \sum_{n=0}^{\infty} q^n \sum_{\substack{\lambda \vdash n \\ \lambda = 1^{\nu_1} 2^{\nu_2} \ldots}} \prod_{\nu_i} \binom{\min(j,\nu_i) - b}{\min(j,\nu_i)}$$

have the same constant and linear term in their coefficients.

Theorem 2 — Proof

From the expansion of HNO_j, we can that the constant term will be $p(n)$.

The linear term can be obtained by observing that each of the linear b will only be multiplied by 1 in the expansion, else they would be a quadratic or higher term, so the linear term is simply the sum of the reciprocal of the squared hooklengths in each partition.

$$\sum_{\lambda \vdash n} \prod_{\substack{h_{ij} \in \lambda \\ h_{ij} \leq j}} (1 - \frac{b}{h_{ij}^2}) = p(n) - \sum_{\substack{\lambda \vdash n \\ h_{ij} \in \lambda \\ h_{ij} \leq j}} \left(\frac{b}{h_{ij}^2} \right) + O(b^2)$$

Theorem 2 — Proof

Now, let's take a look at $C_{1-b,j}$. We can see that for each part frequency in $\lambda \vdash n$, we will get one binomial coefficient, which will be multiplied by all other part frequencies in the same partition. Expanding each of the $\binom{\nu_i - b}{\nu_i}$:

$$\binom{1-b}{1} = 1 - b$$

$$\binom{2-b}{2} = 1 - \frac{3}{2}b + O(b^2)$$

$$\binom{3-b}{3} = 1 - \frac{11}{6}b + O(b^2)$$

$$\dots$$

$$\binom{\nu_i - b}{\nu_i} = 1 - b\sum_{k=1}^{\nu_i}\frac{1}{k} + O(b^2)$$

Theorem 2 — Proof

From this, we can see that the linear term of $\binom{\nu_i - b}{\nu_i}$ is the n^{th} harmonic number, H_n.

The H_n are given by the sum of the reciprocals of the first n positive integers.

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

Theorem 2 — Proof

Notice that if we have a product of two $\binom{\nu_i - b}{\nu_i} * \binom{\nu_i - b}{\nu_i}$, the linear term of their products is going to be the sum of their corresponding harmonic number, $H_{\nu_i} + H_{\nu_i}$.

So, the coefficient on the linear term of $C_{1-b,j}$ will be the following: (For readability, we've omitted the negative in front of the terms, the coefficient we receive is subtracted from the polynomial itself).

$$\sum_{\lambda \vdash n} H_{\min(j,\nu_i)} = \sum_{\substack{\lambda \vdash n \\ \nu_i \geq 1}} 1 + \sum_{\substack{\lambda \vdash n \\ \nu_i \geq 2}} \frac{1}{2} + \sum_{\substack{\lambda \vdash n \\ \nu_i \geq 3}} \frac{1}{3} + \cdots + \sum_{\substack{\lambda \vdash n \\ \nu_i \geq j}} \frac{1}{j}$$

Theorem 2 — Proof

Now, all we must do to show the linear terms match in HNO_j and $C_{1-b,j}$ is to prove the following equality:

$$\sum_{\substack{\lambda \vdash n \\ h_{ij} \in \lambda \\ h_{ij} \leq j}} \frac{1}{h_{ij}^2} \stackrel{?}{=} \sum_{\substack{\lambda \vdash n \\ \nu_i \in \lambda}} H_{\min(j, \nu_j)}$$

We will do this by inducting on j.

Theorem 2 — Proof

Consider 1-Truncations of both series ($j = 1$). This yields

$$\sum_{\substack{\lambda \vdash n \\ h_{ij} \in \lambda \\ h_{ij} = 1}} \frac{1}{1^2} = \sum_{\substack{\lambda \vdash n \\ \nu_i \in \lambda}} H_1.$$

Equivalent, as the number of hooks of size 1 is equal to the number of part sizes in $\lambda \vdash n$.

Theorem 2 — Proof

For up to $(j-1)$-Truncations, we will assume this holds.
For j-Truncations, the added terms will be:

$$\sum_{\substack{\lambda \vdash n \\ h_{ij} \in \lambda \\ h_{ij} = j}} \frac{1}{j^2} = \sum_{\substack{\lambda \vdash n \\ \nu_i \geq j}} \frac{1}{j}$$

Multiplying both sides by j^2 gives

$$\sum_{\substack{\lambda \vdash n \\ h_{ij} \in \lambda \\ h_{ij} = j}} 1 = j \sum_{\substack{\lambda \vdash n \\ \nu_i \geq j}} 1$$

That is: the number of hooks of length j is equal to j times the number of part frequencies j or greater, which is a known result. (Bacher & Manivel) □

Now what?

We've now proven what we set out to prove at the beginning of the semester, so where do we go from here?

Naturally, we decided to look at further b^i and attempt to match them.

Quadratic Equivalence

For $j = 2$, the quadratic and higher b terms of HNO_2 and $C_{1-b,2}$ don't appear to match.
Curiously, it appears that we can add an *incredibly simple* correction term to each b^i to make the two match again.

So, for each polynomial in b on the q^i term of HNO_j and $C_{1-b,j}$:

$$
\begin{aligned}
&(h_0 b^0 + h_1 b^1 + h_2 b^2 + h_3 b^3 + \dots)q^i \\
=&(c_0 b^0 + c_1 b^1 + (c_2 + x_2)b^2 + (c_3 + x_3)b^3 + \dots)q^i
\end{aligned}
$$

respectively, where $(c_k + x_k)b^k = h_k b^k,$ and $c_0 = h_0, c_1 = h_1$

Quadratic Equivalence

Numerically, Keith conjectured that the b^2 terms of the q^i match if we add to the polynomial $\frac{b^2}{16}\nu_4(i)$, where $\nu_4(i)$ is the number of 4s in the partitions of i.

$$\begin{aligned}&(h_0 b^0 + h_1 b^1 + h_2 b^2 + \dots)q^i \\ =&(c_0 b^0 + c_1 b^1 + (c_2 + \frac{1}{16}\nu_4(i))b^2 + \dots)q^i\end{aligned}$$

respectively, where $(c_k + x_k)b^k = h_k b^k,$ and $c_0 = h_0, c_1 = h_1$

Quadratic Equivalence

Where are we in proving it?

Quadratic Equivalence

We started by breaking down the quadratic term of $C_{1-b,2}$:

$$\sum_{\lambda \vdash n} \left(\binom{\gamma_{\geq 1}}{2} + \frac{1}{2}\gamma_{\geq 2} + \frac{3}{2}\gamma_{\geq 2}\gamma_1 + \frac{9}{4}\binom{\gamma_{\geq 2}}{2} \right)$$

Note the following:

$$\begin{aligned} \sum_{\lambda \vdash n} \binom{\gamma_1}{2} &= \sum_{\lambda \vdash n} \frac{1}{2}(\gamma_1^2 - \gamma_1) \\ &= \sum_{\lambda \vdash n} \frac{1}{2}(\gamma_1^2 - (\gamma_{\geq 1} - \gamma_{\geq 2})) \\ &= \sum_{\lambda \vdash n} \frac{1}{2}(\gamma_1^2 - \gamma_{\geq 1} + \gamma_{\geq 2}) \end{aligned}$$

Quadratic Equivalence

Expanding $C_{1-b,2}$ at the quadratic term gives us:

$$\sum_{\lambda \vdash n} \left(\frac{1}{2}\gamma_1^2 - \frac{1}{2}\gamma_{\geq 1} + \frac{1}{2}\gamma_{\geq 2} + \frac{1}{2}\gamma_{\geq 2} + \frac{3}{2}\gamma_{\geq 2}\gamma_{\geq 1} + \frac{9}{8}\gamma_{\geq 2}^2 - \frac{9}{8}\gamma_{\geq 2} \right)$$

Combining $\frac{1}{2}\gamma_1^2 + \frac{3}{2}\gamma_{\geq 2}\gamma_2 + \frac{9}{8}\gamma_{\geq 2}^2$ gives us $\frac{1}{2}(\gamma_1 + \frac{3}{2}\gamma_{\geq 2})^2$. We also combine the $\gamma_{\geq 2}$.

$$\sum_{\lambda \vdash n} \left(\frac{1}{2}(\gamma_1 + \frac{3}{2}\gamma_{\geq 2})^2 - \frac{1}{2}\gamma_{\geq 1} + (\frac{1}{2} + \frac{1}{2} - \frac{9}{8})\gamma_{\geq 2} \right)$$

Quadratic Equivalence

Note that $\frac{1}{2}(\gamma_1 + \frac{3}{2}\gamma_{\geq 2})^2$ can be rearranged as such:

$$\begin{aligned} \frac{1}{2}(\gamma_1 + \frac{3}{2}\gamma_{\geq 2})^2 &= \frac{1}{2}(\gamma_1 + \gamma_{\geq 2} + \frac{1}{2}\gamma_{\geq 2})^2 \\ &= \frac{1}{2}(\gamma_{\geq 1} + \frac{1}{2}\gamma_{\geq 2})^2 \\ &= \frac{1}{2}\gamma_{\geq 1}^2 + \frac{1}{2}\gamma_{\geq 1}\gamma_{\geq 2} + \frac{1}{8}\gamma_{\geq 2}^2 \end{aligned}$$

So, we have:

$$\sum_{\lambda \vdash n} \left(\frac{1}{2}\gamma_{\geq 1}^2 + \frac{1}{2}\gamma_{\geq 1}\gamma_{\geq 2} + \frac{1}{8}\gamma_{\geq 2}^2 - \frac{1}{2}\gamma_{\geq 1} - \frac{1}{8}\gamma_{\geq 2} \right)$$

$$\sum_{\lambda \vdash n} \left(\binom{\gamma_{\geq 1}}{2} + \frac{1}{2}\gamma_{\geq 1}\gamma_{\geq 2} + \frac{1}{8}\gamma_{\geq 2}^2 - \frac{1}{8}\gamma_{\geq 2} \right)$$

Quadratic Equivalence

Let's look at the quadratic term of HNO_2, which should equal the quadratic term of $C_{1-b,2} + \frac{1}{16}\nu_4(n)b^2$.

Let $\mathcal{H}_2(\lambda)$ be the number of hooks of length two in a partition λ.

$$\sum_{\lambda \vdash n} \left(\binom{\ell_0(\lambda)}{2} + \frac{1}{4}\ell_0(\lambda)\mathcal{H}_2(\lambda) + \frac{1}{16}\binom{\mathcal{H}_2(\lambda)}{2} \right) \stackrel{?}{=}$$

$$\sum_{\lambda \vdash n} \left(\binom{\gamma_{\geq 1}}{2} + \frac{1}{2}\gamma_{\geq 1}\gamma_{\geq 2} + \frac{1}{8}\gamma_{\geq 2}^2 - \frac{1}{8}\gamma_{\geq 2} \right) + \frac{1}{16}\nu_4(n)$$

Notice that $\sum_{\lambda \vdash n} \binom{\ell_0(\lambda)}{2}$ and $\sum_{\lambda \vdash n} \binom{\gamma_{\geq 1}}{2}$ count the same thing, so we subtract them from both sides.

Quadratic Equivalence

$$\frac{1}{2}\sum_{\lambda\vdash n}\ell_0(\lambda)\mathcal{H}_2(\lambda)=\sum_{\lambda\vdash n}\gamma_{\geq 1}\gamma_{\geq 2}$$

We have proven these to be equal, and we can use a simple 2:1 bijection to do so, so they cancel.

Quadratic Equivalence

Multiplying both sides by 16 and moving around terms yields the following: Also note that $\sum_{\lambda \vdash n} 2\gamma_{\geq 2} = 2\nu_2(n)$, so we extract that from the sum.

$$\sum_{\lambda \vdash n} \binom{\mathcal{H}_2(\lambda)}{2} + 2\nu_2(n) + \nu_4(n) = 2 \sum_{\lambda \vdash n} \gamma_{\geq 2}^2$$

Quadratic Equivalence

$$\sum_{\lambda \vdash n} \binom{\mathcal{H}_2(\lambda)}{2} + 2\nu_2(n) + \nu_4(n) = 2 \sum_{\lambda \vdash n} \gamma_{\geq 2}^2$$

The generating functions for $\nu_k(n)$ are known and given by Bacher and Manivel. We have the following conjectured generating functions:

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \binom{\mathcal{H}_2(\lambda)}{2} \Bigg) q^n = \frac{1}{(q)_\infty} \left(\frac{q^4 + 3q^6}{(1-q^2)(1-q^4)} \right)$$

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \gamma_{\geq 2}^2 q^n = \frac{1}{(q)_\infty} \frac{q^2(1+q^4)}{(1-q^2)(1-q^4)}$$

Proving these will be sufficient to show that the quadratic terms (with the added term on $C_{1-b,2}$) match.

Quadratic Equivalence

Numerically, this appears to extend to further k, which should be quite fun to prove:

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \gamma_{\geq k}^{2} q^{n} = \frac{1}{(q)_\infty} \frac{q^k(1+q^{2k})}{(1-q^k)(1-q^{2k})}$$

Quadratic Equivalence

We personally proved the generating function for $\binom{\ell_0(\lambda)}{2}$ using theorems from NJ Fine's *Basic Hypergeometric Series and Applications*:

$$\sum_{n=0}^{\infty}\sum_{\lambda\vdash n}\binom{\ell_0(\lambda)}{2}q^n = \frac{1}{(q)_\infty}\frac{q^3}{(1-q)(1-q^2)}$$

So, we believe we can generalize this to $\binom{\gamma_{\geq k}}{2}$ as follows:

$$\sum_{n=0}^{\infty}\sum_{\lambda\vdash n}\binom{\gamma_{\geq k}}{2}q^n = \frac{1}{(q)_\infty}\frac{q^{3k}}{(1-q^k)(1-q^{2k})}$$

OEIS

This work has also resulted in the creation of a few new entries to the Online Encyclopedia of Integer Sequences:
A301313, A302347, and A302348.

As well as another way to obtain A000097.

Further work

Where do we go from here?

Finish proving the b^2 term match.

Further b^i?

Is there a combinatorially interesting polynomial that, when added to $C_{1-b,j}$, is equivalent to HNO_j?

References

- W. J. Keith, "Restricted k-color partitions," *ArXiv e-prints*, Aug. 2014.
- N. J. Fine, "Basic hypergeometric series and applications," no. 27, 1988.
- R. Bacher and L. Manivel, "Hooks and powers of parts in partitions," *Sém. Lothar. Combin.*, vol. 47, pp. Article B47d, 11 pp. (electronic), 2001/02.
- S. Corteel and J. Lovejoy, "Overpartitions," *Trans. Amer. Math. Soc.*, vol. 356, p. 1623, 2004.
- G. Andrews, "Singular overpartitions," *preprint: http://www.personal.psu/gea1/pdf/303.pdf*.