Partitions into a small number of part sizes

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Background: Partitions

A partition of an integer n is a way to write n as a sum of whole numbers, order not mattering. In combinatorial terms from the viewpoint of Gian-Carlo Rota's "Twelvefold Way," it is the number of different distributions of n indistinguishable objects among indistinguishable places.

Here are the 11 partitions of 6:

6	33	22	1111		
51	42	411	3111	2211	21111
321					

Background: Partitions

So we say $p(6) = 11$. Setting by convention $p(0) = 1$, the first few partition numbers are

n	0	1	2	3	4	5	6	7	8	9	10
p(n)	1	1	2	3	5	7	11	15	22	30	42

Euler proved that the *generating function* of the partitions numbers is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

Background: Partitions

There seems to be no immediate reason the number of partitions of n ought to be divisible by anything – no obvious way to group them, for instance – and yet it turns out that as long as $n \equiv 6 \pmod{11}$, $p(n)$ will always be divisible by 11.

This was first observed by the great Indian mathematician Srinivasa Ramanujan in the early 1900s, and proved by him and G. H. Hardy, along with similar theorems:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

Background: Partitions

There are similar congruences $p(An + B) \equiv 0 \pmod{P}$ for every prime $P \geq 5$ and every power of these, and by multiplying the A we can get them for any product of such primes.

However, there is *no* such congruence for $P = 2$ or $P = 3$, and one of the major questions in partition theory is just how the partition function does behave with respect to these two primes.

Background: Partitions

How are identities like this proved? For instance, how can one prove that $p(5n+4) \equiv 0 \pmod{5}$?

One way is to *dissect* the generating function: pick out just the q^{5n+4} terms and ask, is there a neat way to write the generating function for these? Hardy and Ramanujan showed that

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = 5q^4 \prod_{k=1}^{\infty} \frac{(1-q^{25k})^5}{(1-q^{5k})^6}.$$

So all its terms are divisible by 5.

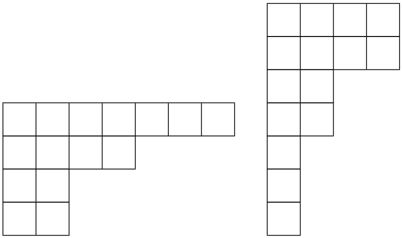
Background: Partitions

Another way is to find a way to group the partitions of $5n+4$. The *rank* of a partition is its largest part minus the number of parts, and Freeman Dyson conjectured (and Atkin and Swinnerton-Dyer showed) that if you group the partitions of $5n+4$ by their residue mod 5, the groups are of equal size.

For instance, the ranks of 4, 31, 22, 211, and 1111 are, respectively, 3, 1, 0, -1, and -3, which are all five residue classes mod 5.

Background: Partitions

If you're looking for identities mod 2, you want a way to pair up partitions, and this is done by *conjugation*. If we stack blocks in rows of height equal to the parts of a partition, we get its *Ferrers diagram*, and one thing we can do to this picture is to reflect it across the main diagonal. If the results are different from each other, those two partitions are conjugate to each other, while a partition fixed under this map is *self-conjugate*.



Background: Partitions

Another type of theorem in partitions observes structure or symmetry that holds for subsets of the partitions of n that satisfy some conditions. Euler was the first to note the most famous theorem of this type: the number of partitions of n in which only odd sizes are allowed, equals the number of partitions of n in which only one part of any given size can appear. For $n = 6$, there are 4 of each of these.

6	33	22	1111		
51	42	411	3111	2211	21111
321					

Background: Partitions

The restriction we'll be interested in today is demanding that our partitions have a certain number of sizes of part. Let $\nu_i(n)$ be the number of partitions of n that have exactly i different part sizes.

6	33	22	1111		
51	42	411	3111	2211	21111
321					

The part sizes are already sorted in rows, so we can see that $\nu_1(6) = 4$, $\nu_2(6) = 6$, and $\nu_3(6) = 1$.

Other than intellectual curiosity, is there any particular reason to be interested in these?

Background: Overpartitions

Partitions can model many physical systems (those with indistiguishable parts and places), but if we want more structure, we can add additional details. For instance, we can talk about partitions with multiple “colors” of part, which correspond to vectors of partitions which total to n.

Very popular in recent years are *overpartitions*, which are partitions into two colors of part where only one color can be used per size of part, usually denoted by overlining the last instance of that part size if the second color is used.

Background: Overpartitions

The overpartitions of 4 are

4	$\overline{4}$	22	$2\overline{2}$	1111	$111\overline{1}$		
31	$\overline{3}1$	$3\overline{1}$	$\overline{3}\overline{1}$	211	$\overline{2}11$	$21\overline{1}$	$\overline{2}1\overline{1}$

So there are 14 overpartitions of 4.

Background: Overpartitions

How interested are people in overpartitions? They were invented in this paper:

Overpartitions
S **Corteel**, J **Lovejoy** - Transactions of the American Mathematical Society, 2004 - ams.org
Abstract: We discuss a generalization of partitions, called overpartitions, which have proven useful in several combinatorial studies of basic hypergeometric series. After showing how a number of finite products occurring in $ q $-series have natural interpretations in terms of ...
Cited by 174 Related articles All 5 versions Cite Save

Background: Overpartitions

Since every part size can have one of two colors, it's easy to see that the number of overpartitions of n, $\overline{p}(n)$, satisfies

$$\overline{p}(n) = 2\nu_1(n) + 4\nu_2(n) + 8\nu_3(n) + \dots.$$

So knowing something about the $\nu_i(n)$ can tell you something about $\overline{p}(n)$.

(But really I just thought it was a neat question.)

ν_1

The first case is easy. We have that $\nu_1(n)$ is just $d(n)$, the number of divisors of n. This function is pretty completely understood. If the prime factorization of n is

$$n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r},$$

then

$$d(n) = (e_1 + 1)(e_2 + 1) \dots (e_r + 1).$$

So the first interesting case is ν_2.

Theorem

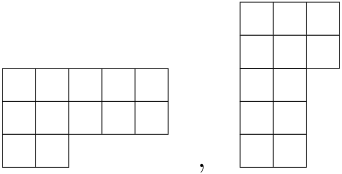
If $m \equiv 2 \pmod 4$, or if two or more primes appear to odd order in the prime factorization of m, them $\nu_2(m) \equiv 0 \pmod 2$.

To prove this, we are going to pair up partitions. Note that the number of distinct part sizes of a partition is invariant under conjugation, and the parity of a set of partitions closed under conjugation is equal to the parity of its self-conjugate subset.

The theorem will be proved if we guarantee that any possible partitions can be paired up, or at least that there are an even number of exceptions to the pairing.

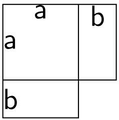
ν_2

The first thing to pair up is any partition which has exactly two part sizes but is not self-conjugate.

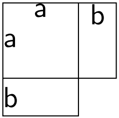


ν_2

Any self-conjugate partition into two sizes of part has a Ferrers diagram that looks like this:



In other words, $m = (a+b)^2 - b^2$. But it is elementary that $4n+2$ is not a difference of squares, so no such partition can exist. That covers the $m = 4n+2$ case.



For the more general m ("two primes to odd order in its factorization"), note that we can also say $m = a^2 + 2ab = a(a + 2b)$. We know a has to be the of the parity of m. So any a will give a b as long as $a|m$, $a < \sqrt{m}$, and either

$$0 < s_2(a) < s_2(m) \text{ or } s_2(a) = 0 = s_2(m).$$

Here $s_p(m)$ is the order of p in the factorization of m.

ν_2

Let $m = {p_a}^{\alpha_0} {p_b}^{\alpha_1} p_1^{\beta_1} \dots p_r^{\beta_r}$, with α_0 and α_1 odd. If p_a or p_b is 2, the relevant exponent is at least 2.

If m is odd, every $a|m$, $a < \sqrt{m}$ gives a suitable partition, and since $d(n) \equiv 0 \pmod 4$, the half below $\sqrt{m}$ are of even number.

If m is even, let $m' = \frac{m}{4}$. For any suitable a, let $a = 2c$, so $m = 2c(2c + 2b)$, so $m' = c(c + b)$. Now any divisor $c|m'$ below $\sqrt{m'}$ gives a suitable partition, and again we have an even number of these.

$\square$

ν_2

This gives us, among others,

Theorem

$$\begin{aligned}
\nu_2(4n+2) &\equiv 0 \pmod{2} \\
\nu_2(9n+6) &\equiv 0 \pmod{2} \\
\nu_2(25n+10) &\equiv 0 \pmod{2} \\
\nu_2(25n+15) &\equiv 0 \pmod{2} \\
&\cdots
\end{aligned}$$

Any arithmetic progression $p(pn+r)$ with r a quadratic nonresidue mod a prime p will do.

ν_2 and ν_1

Remember that

$$\overline{p}(m) = 2\nu_1(m) + 4\nu_2(m) + \dots.$$

Except for the $m = 4n + 2$ case, all of our progressions here have $d(p(pn + r)) \equiv 0 \pmod 4$, because two of the $(e_i + 1)$ will be even. Since $\nu_1(m) = d(m)$, we have $\nu_1(m) \equiv 0 \pmod 4$ and $\nu_2(m) \equiv 0 \pmod 2$, so we now know

Theorem

In any of these progressions other than $m = 4n + 2$, $\overline{p}(m) \equiv 0 \pmod 8$.

ν_2

A note to emphasize for later observation: other than $m = 4n + 2$, all of the progressions where we were just now able to prove that $\nu_2(An + B) \equiv 0 \pmod 2$ rely on having two primes to odd power in the factorizations of $An + B$: that means that $\nu_2(An + B) \equiv 0 \pmod 2$ most of the time means that $\nu_1(An + B) \equiv 0 \pmod 4$.

Are the ν_i related in such progressions?

ν_2

There are two ways to go further from here. We could start looking at ν_3, which would let us say things modulo 16 about overpartitions, or we could see if something can be said about ν_2 modulo higher powers of 2.

ν_2

Theorem

The following are all divisible by 4 for all $n \geq 0$: $\nu_2(16n + 14)$, $\nu_2(36n + 30)$, $\nu_2(72n + 42)$, $\nu_2(196n + 70)$, $\nu_2(252n + 114)$.

Notice that all of these progressions are all of numbers 2 mod 4: they have no self-conjugate partitions into two part sizes, so all their partitions counted by ν_2 come in conjugate pairs.

It would be really nice if we could prove this theorem in the same hands-on way, by pairing up conjugate pairs to show that these are divisible by 4. I haven't been able to figure out how to to that yet...

Instead, the proof will rely on a different kind of technique in partition theory: *modular forms*. First, we need to describe ν_2 in more detail.

ν_2

Let $d(n)$ be the number of divisors of n and $\sigma_1(n) = \sum_{d|n} d$, the sum of all the divisors of n. We have the following identity for $\nu_2(n)$:

$$\nu_2(n) = \frac{1}{2}\left(\sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n)\right).$$

Think of taking all possible pairs of rectangles of parts to make a partition. One will have "area" k, and the other will have "area" $n-k$. For each k, there are $d(k)$ rectangles (the heights have to divide k).

ν_2

$$\nu_2(n) = \frac{1}{2}\left(\sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n)\right).$$

Sometimes, the two heights will be the same, and in that case we don't get a partition into two different sizes. This happens when the height is any divisor j of n, and the width of the first rectangle is anything from 1 to n/j.

So if we subtract off $\sigma_1(n)$ to account for these, we've subtracted off 1 too many for each divisor, so we add those back in. Now we have two distinct heights, and we take half the number of such pairs to account for just taking those in which the first rectangle is of greater height.

ν_2

Let's prove our theorem for $16n + 14$.

$$\nu_2(n) = \frac{1}{2}\left(\sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n)\right).$$

First, $\sigma_1(16n + 14) \equiv 0 \pmod 8$, because $16n + 14 = 2(8n + 7)$, and $8n + 7$ has exactly one of either

1. an odd number of primes $8j + 7$ to odd power in its prime factorization, or
2. an odd number each of primes $8j + 3$ and $8j + 5$ to odd powers.

If it's the first case, say $k = 8j + 7$ is such a prime, and then we can group divisors by whether they have 2 or k: $x + 2x + kx + 2kx = x(3 + 3k) = 3x(1 + k)$, and $1 + k \equiv 0 \pmod 8$. A similar (longer) identity holds for the other case.

So we can just throw $-\frac{1}{2}\sigma_1(n)$ out of the calculation; it is 0 mod 4.

$$\nu_2(16n+14) = \frac{1}{2}\left(\sum_{k=1}^{16n-1} d(k)d(n-k) - \sigma_1(n) + d(n)\right)$$

We also have $d(16n+14) \equiv d(8n+7)^2 \equiv 0 \pmod 4$, so we can throw out these two terms as well.

Why? Because $8n+7$ has at least 1 prime to odd order, and then $16n+14$ has at least 2: that makes $d(8n+7)$ at least 2 and possibly 4 modulo 4, and $16n+14$ always twice that.

Now all the divisors of $m = 16n + 14$ above and below the halfway mark can be paired off in the sum. That leaves us with

$$\nu_2(16n + 14) \equiv \sum_{k=1}^{8n+6} d(k)d(m-k) \pmod 4.$$

So now we want to show that this is 0 mod 4. (So far all of this logic applies to all of the progressions in the theorem.)

For the method to do this I thank a colleague by the name of Jeremy Rouse.

If k is square, $d(k)$ is odd; if k has 1 prime to power 1 mod 4, $d(k)$ is 2 mod 4, and otherwise it is 0 mod 4. Furthermore, $16n+14$ cannot be the sum of two squares.

$$\nu_2(16n+14) \equiv \sum_{k=1}^{8n+6} d(k)d(m-k) \pmod 4$$

So the only terms that are not multiples of 4 are those in which k or $m-k$ is square, and the other term is 2 mod 4, i.e. $m-k$ or k respectively is py^2 for p a prime, with $s_p(y) \equiv 0 \pmod 2$. Since $m \equiv 6 \pmod 8$, when $m = x^2 + py^2$ either x is even, $p = 2$ and y is odd, or x is odd, y is odd and $p \equiv 5 \pmod 8$.

Build the following two functions:

$$F(q) := \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1} \equiv \sum_{n=1}^{\infty} q^{(2n+1)^2} \pmod 2$$

and

$$G(q) := \frac{1}{2} \sum_{n=0}^{\infty} \sigma_1(8n+5) q^{8n+5} \equiv \sum_{\substack{p \equiv 5 \pmod 8 \\ y \geq 1, 2 | s_p(y)}} q^{py^2} \pmod 2.$$

With these functions,

$$T(q) = F(q)G(q) + F(q^4)F(q^2)$$

has coefficient of q^{16n+14} of parity equal to the number of representations $x^2 + py^2$ of $16n+14$ that we desire.

ν_2

I will not go into detail here about how this works, but the key idea is that $T(q)$ is a *modular form*, a function for which there is a finite procedure to check the parity of all of its coefficients.

We run this check and we find that all the coefficients are even, so there are an even number of representations of $16n + 14$ of the form $x^2 + py^2$ with $s_p(y)$ even. (For instance, for 14 these happen to be $1^2 + 13 \cdot 1^2$ and $3^2 + 5 \cdot 1^2$.) Each contributes 2 to the sum, so the total is 0 mod 4. $\square$

We need more terms in the analogue of $T(q)$ for the other progressions because there are more possible types of representations, but once we have constructed the relevant forms the same check procedure works.

For ν_3, we will go backwards in a way: from information about overpartitions to information about ν_3.

Theorem

$\nu_3(An + B) \equiv 0 \pmod{2}$ for $(A, B) = (36, 30), (72, 42), (196, 70)$, and $(252, 114)$.

In each of these progressions we have already shown that $\nu_1(n) \equiv 0 \pmod{8}$ and $\nu_2(n) \equiv 0 \pmod{4}$. If we can show that $\overline{p}(n) \equiv 0 \pmod{16}$, then it must be the case that $\nu_3(n) \equiv 0 \pmod{2}$.

To prove these, we will use a third technique from partition theory: *generating function dissection.*

The dissection of a generating function is a way of writing it that separates out all the terms with a given residue for some modulus. We saw it earlier when we said that

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = 5q^4 \prod_{k=1}^{\infty} \frac{(1-q^{25k})^5}{(1-q^{5k})^6} = 5q^4 \frac{f_{25}^5}{f_5^6},$$

where we shorten notation with $f_i = \prod_{k=1}^{\infty}(1-q^{ik})$.

The generating function for overpartitions is

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{k=1}^{\infty} \frac{1+q^k}{1-q^k} = \prod_{k=1}^{\infty} \frac{1-q^{2k}}{(1-q^k)^2} = \frac{f_2}{f_1{}^2}.$$

The following dissection was proved by Hirschhorn and Sellers:

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q\frac{f_6^2 f_{18}^3}{f_3^6}.$$

If we are interested in $\overline{p}(36n+30)$, we only need the first term:

$$\sum_{n=0}^{\infty} \overline{p}(3n)q^{3n} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3}$$

or

$$\sum_{n=0}^{\infty} \overline{p}(3n)q^{n} = \frac{f_2^4 f_3^6}{f_1^8 f_6^3}.$$

Our process will be to repeatedly extract progressions we are interested in, and dissect again to get finer progressions, until we reach 36n+30. Our goal will be to show that all coefficients in the final progression are 0 mod 16.

We also need the following identities, which were proved by Ramanujan:

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}.$$

We saw that

$$\sum_{n=0}^{\infty} \overline{p}(3n)q^n = \frac{f_2^4 f_3^6}{f_1^8 f_6^3}.$$

If we want to dissect this mod 2, we can factor out the even powers:

$$\sum_{n=0}^{\infty} \overline{p}(3n)q^n = \left(\frac{f_2^4}{f_6^3}\right) \frac{f_3^6}{f_1^8}$$

$$= \left(\frac{f_2^4}{f_6^3}\right) \left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3 \frac{f_6 f_{48}^2}{f_{24}}\right)^3 \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}\right)^2.$$

We expand out all the powers and throw out anything with an odd q-power on the front. The even terms out of $\sum \overline{p}(3n)q^n$ will be $\sum \overline{p}(6n)q^{2n}$. We can also throw out any term with a coefficient divisible by 16 or more. We end up with

$$\sum \overline{p}(6n)q^n \equiv \frac{f_2^4 f_{12}^{15}}{f_1^8 f_6^6 f_{24}^6} + 12q^3 \frac{f_{12}^3 f_{24}^2}{f_6^2} \pmod{16}.$$

But all powers in the second term are of the form $6n+3$, so they are irrelevant to us.

Since we are interested in $36n+30$, we focus on the first term:

$$\sum_{n=0}^{\infty} \overline{p}(6n)q^n \equiv 12q^3 \frac{f_2^4 f_{12}^{15}}{f_1^8 f_6^6 f_{24}^6} + \dots$$
$$\equiv \frac{f_{12}^{15}}{f_6^6 f_{24}^6} \left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q \frac{f_6^2 f_{18}^3}{f_3^6} \right)^4 + \dots \pmod{16}$$

Expand out and pick the terms of the form q^{3n+2} to get

$$\sum_{n=0}^{\infty} \overline{p}(18n+12)q^{3n+2} \equiv \frac{f_{12}^{15}}{f_6^6 f_{24}^6} \left(24q^2 \frac{f_6^7 f_9^{18}}{f_3^{30} f_{18}^6} \right) \pmod{16}.$$

$$\sum_{n=0}^{\infty} \overline{p}(18n+12)q^{3n+2} \equiv \frac{f_{12}^{15}}{f_6^6 f_{24}^6}\left(24q^2 \frac{f_6^7 f_9^{18}}{f_3^{30} f_{18}^6}\right) \pmod{16}.$$

Out of this we want the terms for odd n. But $(1-q^i)^2 = 1 - 2q^i + q^{2i} \equiv 1 - q^{2i} \pmod 2$, so $f_i^2 \equiv f_{2i} \pmod 2$ and $24 f_i^2 \equiv f_{2i} \pmod{16}$. With that, we can say

$$24\frac{1}{f_3^{30}} \equiv 24\frac{1}{f_6^{15}} \pmod{16}.$$

So

$$\sum_{n=0}^{\infty} \overline{p}(18n+12)q^{3n+2} \equiv \frac{f_{12}^{15}}{f_6^6 f_{24}^6} \left(24q^2 \frac{f_6^7 f_9^{18}}{f_6^{15} f_{18}^6} \right) \pmod{16}.$$

But this does not have any odd n terms. That means no term of the form q^{36n+30} has a coefficient which is nonzero mod 16.

Hence $\overline{p}(36n+30) \equiv 0 \pmod{16}$.

And since $\nu_1(36n+30) \equiv 0 \pmod{8}$ and $\nu_2(36n+30) \equiv 0 \pmod{4}$, it must follow that $\nu_3(36n+30) \equiv 0 \pmod{2}$. $\square$

Open Questions

Our proof for all our ν_2 being 0 mod 4 used progressions where $d(An+B) \equiv 0 \pmod 8$, $d((An+B)/2) \equiv 0 \pmod 4$, and $An+B$ is never the sum of two squares. I haven't found any progression where this is not the case:

Conjecture

The stated conditions are necessary for $\nu_2(An+B) \equiv 0 \pmod 4$ *for all* n.

Likewise, our proof for $\nu_3(An+B) \equiv 0 \pmod 2$ required $\nu_2(An+B) \equiv 0 \pmod 4$. Again, is this a necessary condition?

Conjecture

If $\nu_3(An+B) \equiv 0 \pmod 2$ *for all* n, *then* $\nu_2(An+B) \equiv 0 \pmod 4$ *for all* n.

Open Questions

We know $16n + 14$ has no self-conjugate partitions into 2 sizes of part, because $16n + 14 \equiv 2 \pmod 4$. Can we prove $\nu_2(16n + 14) \equiv 0 \pmod 4$ more combinatorially, by finding a matching that pairs up pairs of conjugate partitions?

It is known that there are progressions in $\overline{p}$ that are 0 modulo any power of 2: for instance, $\overline{p}(72n + 69) \equiv 0 \pmod{32}$. Congruences also hold for other primes: for instance, $\overline{p}(27n + 19) \equiv 0 \pmod 3$. But so far I have found no examples of the following:

1. Progressions for ν_2 or ν_3 of any modulus other than 2 or 4;
2. Progressions for any ν_i for $i > 3$.

Do these exist? If so, what are some, and how can they be found? If they don't exist, why not?