Incidence structures, codes, and Galois geometry

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Incidence Structures

A finite **incidence structure** D=($\mathcal{X}, \mathcal{B}$) is a finite set $\mathcal{X}$ of *points* and a collection $\mathcal{B}$ of subsets called *blocks*.

An incidence structure is simple if all blocks are distinct as sets.

The complementary structure D^* of an incidence structure D has as blocks the complements of the blocks of D.

Design

A t-(v, k, λ) design is an incidence structure $\mathcal{D}$=($\mathcal{X}, \mathcal{B}$) such that:

- $|\mathcal{X}| = v$,
- $|B| = k$ for each $B \in \mathcal{B}$, and
- Every t-subset of $\mathcal{X}$ s contained in exactly λ blocks.

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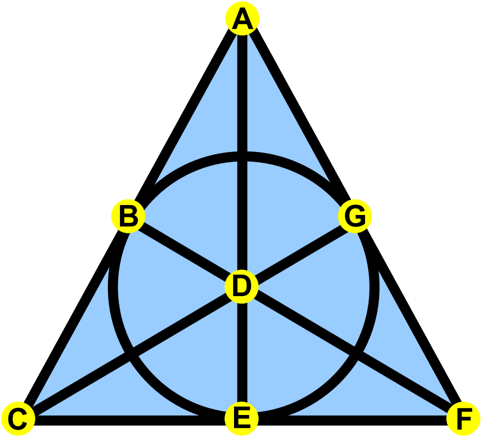

A 2-$(7, 3, 1)$ design

Incidence Matrix

The incidence matrix of a t-(v, k, λ) design is a $b \times v$ $(0, 1)$ matrix whose (i, j) entry is 1 if block i contains point j, and 0 otherwise.

The 2-(7,3,1) Design:

	A	B	C	D	E	F	G
B_1	1	1	1	0	0	0	0
B_2	1	0	0	1	1	0	0
B_3	1	0	0	0	0	1	1
B_4	0	1	0	1	0	1	0
B_5	0	1	0	0	1	0	1
B_6	0	0	1	1	0	0	1
B_7	0	0	1	0	1	1	0

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Finite Geometries

Projective Geometry $PG(n, q)$

- **points** of $PG(n, q)$ are the 1-dimensional subspaces of $\mathbb{F}_q^{n+1}$.
- **lines** of $PG(n, q)$ are the 2-dimensional subspaces of $\mathbb{F}_q^{n+1}$.
- d-dimensional **projective** subspaces are the $(d+1)$-dimensional subspaces of $\mathbb{F}_q^{n+1}$.

Affine Geometry $AG(n, q)$

- **points** of $AG(n, q)$ are the vectors of $\mathbb{F}_q^n$.
- **lines** of $AG(n, q)$ are the 1-dimensional subspaces of $\mathbb{F}_q^n$ and their cosets.
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The geometric designs

A geometric design is formed from the points and d-subspaces of $PG(n,q)$ or $AG(n,q)$.

The projective geometry design $PG_d(n,q)$:

$$2-\left(\frac{q^{n+1}-1}{q-1}, \frac{q^{d+1}-1}{q-1}, \frac{(q^{n+1}-q^2)(q^{n+1}-q^3)\cdots(q^{n+1}-q^d)}{(q^{d+1}-q^2)(q^{d+1}-q^3)\cdots(q^{d+1}-q^d)}\right)$$

The affine geometry design $AG_d(n,q)$:

$$2-\left(q^n, q^d, \frac{(q^n-q)(q^n-q^2)\cdots(q^n-q^{d-1})}{(q^d-q)(q^d-q^2)\cdots(q^d-q^{d-1})}\right)$$

If $q=2$, $AG_d(n,2)$ is also a 3-$(2^n, 2^d, \frac{(2^n-2^2)\cdots(2^n-2^{d-1})}{(2^d-2^2)\cdots(2^d-2^{d-1})})$ design.

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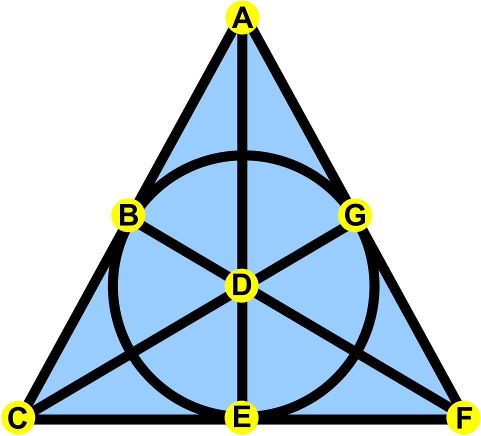
A small example: $PG_1(2,2)$



$PG_1(2,2)$: The projective plane of order 2

Linear error-correcting codes

Linear code

A **linear** q**-ary** $[n, k, d]$ **code** C is a k-dimensional subspace of the n-dimensional vector space over the field $GF(q)$ of order q with minimum Hamming distance d.
A code with **minimum distance** d can **correct** up to $e = [(d-1)/2]$ errors.

Dual code

The dual code $C^{\perp}$ of an $[n, k]$ code C is the $[n, n-k]$ code defined by

$$C^{\perp} = \{y \in GF(q)^n \mid y \cdot x = 0 \text{ for all } x \in C\}$$

Parity check matrix

A matrix H of q-rank $n-k$ whose rows are vectors from $C^{\perp}$ is a parity check matrix of C.

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If a codeword $x = (x_1, \ldots, x_n) \in C$ is sent over a communication channel, and a vector $y = (y_1, \ldots, y_n)$ is received, for each coordinate i, $1 \leq i \leq n$, the values

$$y_i^{(1)}, \ldots, y_i^{(r_i)} \tag{1}$$

of r_i linear functions are computed, and y_i is decoded as the most frequent among the values (1).

Theorem. (Rudolph, 1967)

If C is a linear $[n, k]$ code such that $C^{\perp}$ contains a set **S** of vectors of weight w whose supports are the blocks of a 2-(n, w, λ) design, the code C can correct up to

$$e = \left[\frac{r + \lambda - 1}{2\lambda} \right]$$

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If a codeword $x = (x_1, \ldots, x_n) \in C$ is sent over a communication channel, and a vector $y = (y_1, \ldots, y_n)$ is received, for each coordinate i, $1 \leq i \leq n$, the values

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of r_i linear functions are computed, and y_i is decoded as the most frequent among the values (1).

Theorem. (Rudolph, 1967)

If C is a linear $[n, k]$ code such that $C^{\perp}$ contains a set **S** of vectors of weight w whose supports are the blocks of a 2-(n, w, λ) design, the code C can correct up to

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Which codes support t-designs?

Task

Find a linear code C such that $C^\perp$ supports a t-design with $t \geq 2$.

A construction from incidence matrices

If C is a linear code over $GF(q)$ of length v with a parity check matrix H being the $b \times v$ incidence matrix of a t-(v, w, λ) design D, then $C^\perp$ supports the t-(v, w, λ) design D.

The dimension of C is

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Two fundamental questions

Given parameters $v > w > 0$, $\lambda > 0$, such that a 2-(v, w, λ) design exists,

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Theorem. (Hamada '73)

(a) The p-rank of $PG_d(n, p^s)$ is given by

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Hamada's Conjecture

Conjecture (Hamada, 1973)

A geometric design, $PG_d(n, q)$ or $AG_d(n, q)$, $(q = p^t)$, has minimum p-rank among all designs with the given parameters.

Note

The number of nonisomorphic designs having the same parameters as geometric designs **grows exponentially**: Jungnickel '84, Kantor '94, Lam, Lam & T '00, '02, Jungnickel & T, '09, Clark, Jungnickel & T, 09.

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Implications

- Majority logic decodable codes: Hamada’s conjecture indicates that geometric designs are the best choice for the given design parameters.
- Uniqueness: The conjecture provides a simple characterization of the geometric designs. Finding isomorphisms is a computational problem with exponential complexity, while calculating p-rank is done in polynomial time.

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Proved cases

Hamada's Conjecture has been proved in the following cases:

- **Hamada and Ohmori (1975):** True for $PG_{n-1}(n,2)$ and $AG_{n-1}(n,2)$.
- **Doyen, Hubaut, Vandensavel (1978):** True for $PG_1(n,2)$ and $AG_1(n,3)$.
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Note

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Are geometric designs characterized by their p-rank?

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Are geometric designs characterized as the **unique** (up to isomorphism) designs with the given parameters and p-rank?

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Yes, in all proved cases of Hamada's Conjecture.

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Non-geometric designs with the same p-rank as geometric ones

- A 2-$(31, 7, 7)$ design supported by the binary quadratic-residue code (Goethals and Delsarte '68).
- All (five) quasi-symmetric 2-$(31, 7, 7)$ $(x = 1, y = 3)$ designs, one being $PG_2(4, 2)$: (Tonchev '86).
- All (five) 3-$(32, 8, 7)$ designs with even block intersection numbers, one being $AG_3(5, 2)$: (Tonchev '86).
- Two affine-resolvable 2-$(64, 16, 5)$ designs with the parameters of $AG_2(4, 3)$: (Harada, Lam and Tonchev '05).
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- An infinite class of designs with the parameters of $PG_d(2d, p)$, where $d \geq 2$ and p is any prime: (Jungnickel and Tonchev, 2008).
- An infinite class of designs with the parameters of $AG_{d+1}(2d + 1, 2)$ for every $d \geq 2$: (Clark, Jungnickel and Tonchev, 2011).

Revising the Hamada conjecture

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A generalized incidence matrix of a design has entries in $GF(q)$, with nonzero elements of $GF(q)$ designating incidence.

Definition (T '99)

The dimension of a design D over $GF(q)$, ($dim_q(D)$), is defined as the minimum q-rank of all generalized incidence matrices of D over $GF(q)$.

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Let q be an arbitrary prime power, and let $n \geq 2$.
(i) Let D be a 2-$((q^{n+1}-1)/(q-1), q^n, q^n - q^{n-1})$ design. Then

$$dim_q(D) \geq n + 1.$$

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What is the dimension of $PG_d(n, q)$ for $d < n - 1$?

The weight distribution of the binary $[15, 4]$ simplex code $S_2(4)$:

$$A_0 = 1, \; A_8 = 15.$$

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The codewords of weight 8 support complements of **planes** in $PG(3, 2)$.

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Theorem (Jurrius '12).

- The codewords of weight $(q^{n+1} - q^{d+1})/(q - 1)$ of the linear code over $GF(q^d)$ spanned by the q-ary simplex code $S_q(n)$ support the complements of $(n - d)$-subspaces of $PG(n, q)$.
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Subfield, trace, and Galois closed codes

Let C be a linear code over $E = GF(q^t)$, being a field extension of $F = GF(q)$.

Subfield subcode C_F

The subfield subcode $C_F \subseteq C$:

$$C_F = \{x = (x_1, \ldots, x_n) \mid x \in C,\ x_i \in GF(q)\}$$

Trace code $Tr(C)$

$$Tr(C) = \{\ Tr(x) = (Tr(x_1), \ldots, Tr(x_n))\ | x \in C,\ \}$$

where $Tr = Tr_{E/F}$:

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$$(Tr(C))^\perp = (C^\perp)_F.$$

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We define the q-dimension of D^*, $dim_q D^*$, as the smallest dimension of any $GF(q)$-**trace code** $Tr(C(M))$, where E runs over all finite **extension fields** of $GF(q)$, and M runs over all generalized E-incidence matrices of D^*.

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Embeddings and the q-dimension

Definition (Jungnickel and Tonchev '13).

Let $D = (V, \mathcal{B})$ be a simple incidence structure.
D is embedded in $\Pi = PG(N, q)$ if V corresponds to a set of points of Π, and every block B is the intersection of V with some projective subspace W of Π:

$$B = V \cap W.$$

Theorem (Jungnickel and Tonchev '13)

Let D be a simple incidence structure, and q be a prime power.
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Let $\mathcal{D}$ be a simple incidence structure which is embedded in $\Pi = PG(N-1, q)$, and let $E = GF(q^t)$ be any extension field of $F = GF(q)$ satisfying

$$t \geq n - d,$$

where d is the smallest dimension of a subspace of Π associated with some block of $\mathcal{D}$.

Then there exists a generalized E-incidence matrix M for the complementary incidence structure $\mathcal{D}^*$ of $\mathcal{D}$ such that the trace code $Tr(C(M))$ associated with M has dimension at most N over F.

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A characterization of $PG_d(n,q)$ and $AG_d(n,q)$

Theorem (Jungnickel and Tonchev '13)

1. Let D^* be the complementary design of a design D having the **same parameters** as $PG_d(n,q)$, $1 \le d \le n-1$. Then

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Prove Part 2 for all d in the range $2 \le d \le (n-2)/2$.

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A characterization of $PG_d(n, q)$ and $AG_d(n, q)$

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1. Let D^* be the complementary design of a design D having the **same parameters** as $PG_d(n, q)$, $1 \leq d \leq n-1$. Then
$$\dim_q(D^*) \geq n+1,$$
and $\dim_q(D^*) = n+1$ if and only if D is isomorphic to $PG_d(n, q)$.
2. Let D^* be the complementary design of a design D having the **same parameters** as $AG_d(n, q)$, where $d = 1$ or $(n-2)/2 < d \leq n-1$. Then
$$\dim_q(D^*) \geq n+1,$$
and $\dim_q(D^*) = n+1$ if and only if D is isomorphic to $AG_d(n, q)$.

Open Problem

Prove Part 2 for all d in the range $2 \leq d \leq (n-2)/2$.

More examples

Theorem (J & T '13)

The q-dimension of the classical 3-$(q^2+1, q+1, 1)$ design being the Möbius (or Miquelian) plane of order q, is equal to 4.

Theorem (J & T '13)

The dimension of any 2-$(2^s(2^t-2^{t-s}+1, 2^s, 1)$ design over $GF(2^t)$, being a maximal 2^s-arc $(1 \leq s \leq t-1)$ in $PG(2, 2^t)$, is equal to 3.

Theorem (J & T '13)

The q-dimension of a 2-$(q^3+1, q+1, 1)$ design being the Hermitian unital in $PG(2, q^2)$, is equal to 3.

More examples

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Thank You!

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Any Questions?

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