An Introduction to Partition Theory, Part I

Spring 2015

Michigan Tech Combinatorics Seminar

January 29, 2015

Textbook

Recommended text: Andrews, George. *The Theory of Partitions.* Encyclopedia of Mathematics and its Applications, Vol. 2. G.-C. Rota, ed. 1976. ISBN 0-521-63766-X.

Alternative undergraduate text: Andrews, George, and Eriksson, Kimmo. *Integer Partitions.* Cambridge University Press, 2004. ISBN 0-521-84118-6.

Basic Definitions

Definition

A *partition* of a nonnegative integer n is a nonincreasing sequence of positive integers with sum n. We say $\lambda \vdash n$, "λ partitions n."

We write partitions as sums, sequences, or occasionally with the *frequency notation*. Here are the partitions of 4:

4	3+1	2+2	2+1+1	1+1+1+1
(4)	(3,1)	(2,2)	(2,1,1)	(1,1,1,1)
4^1	$3^1 1^1$	2^2	$2^1 1^2$	1^4

The counting function for the number of partitions of n is usually denoted $p(n)$, so $p(4) = 5$.

Basic Definitions

We'll speak more about the connections between partition theory and other areas of mathematics later in the series, but for the moment, here is one motivation:

(1234)	(123)(4)	(12)(34)	(12)(3)(4)	(1)(2)(3)(4)
(1243)	(132)(4)	(13)(24)	(13)(2)(4)	
(1324)	(124)(3)	(14)(23)	(14)(2)(3)	
(1342)	(142)(3)		(23)(1)(4)	
(1423)	(134)(2)		(24)(1)(3)	
(1432)	(143)(2)		(34)(1)(2)	
	(234)(1)			
	(243)(1)			

Each column is a conjugacy class of permutations in S_4; each conjugacy class is described by the partition which constitutes its cycle lengths. So enumerating these is useful.

Basic Definitions

Recall that, given a sequence $a(n)$ defined on $\mathbb{N}_0$, its *generating function* is the power series $\sum_{n=0}^{\infty} a(n)q^n$. For the partition function $p(n)$, the generating function is

Theorem

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

Proof.

Since $\frac{1}{1-q^k} = 1 + q^k + q^{2k} + \dots$, the product

$$(1 + q + q^2 + q^3 + \dots)(1 + q^2 + q^4 + q^6 + \dots)\dots$$

gives rise to a term q^n once for each selection of frequencies of parts $1, 2, 3, \dots$ that gives a partition of n. $\square$

Basic Definitions

Form the method of proof, it is easy to see that we can consider restricted classes of partitions based on the sizes of parts that appear, or the number of times they appear. For instance, partitions in which all parts are odd have generating function

$$\mathcal{O}(q) = \sum_{n=0}^{\infty} p_o(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1 - q^{2k-1}}.$$

Partitions in which parts may appear at most once have generating function

$$\mathcal{D}(q) = \sum_{n=0}^{\infty} p_d(n) q^n = \prod_{k=1}^{\infty} (1 + q^k).$$

Basic Definitions

Many classical theorems in partition theory state identities between such classes which would not be obvious from a casual inspection. The usual first such theorem is

Theorem

$\mathcal{O}(q) = \mathcal{D}(q)$. *That is, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

Remark: In fact this was roughly *the* first theorem in partition theory, proved by Leonhard Euler in his work *De Partitio Numerorum*, which first systematically explored the concept.

We shall prove this simple theorem in several ways, each illustrating a method of proof in partition theory.

Proof Methods

Proof Number 1: generating function manipulation.

Proof.

$$\mathcal{O}(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}} = \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^{k}} = \prod_{k=1}^{\infty} (1+q^{k}) = \mathcal{D}(q).$$

□

Generating functions are strictly formal power series, so we almost never care about questions of convergence.

Exercise 1 (from Andrews)

Prove using generating functions: the number of partitions of n in which parts may appear 2, 3, or 5 times is equal to the number of partitions of n into parts 2, 3, 6, 9, or 10 mod 12.

Euler on convergence

Regarding convergence, Euler couldn't have cared less. He considered "the number of 'partitions' of $n \in \mathbb{Z}$ into positive or negative powers of 3," which has generating function

$$\prod_{k=1}^{\infty}(q^{-3^k} + 1 + q^{3^k}) = \cdots + q^{-2} + q^{-1} + 1 + q^1 + q^2 + q^3 + \ldots .$$

Of course, this is just the neg-trinary representation of n, so all the coefficients are 1 and are combinatorially perfectly well-defined, but the power series itself converges nowhere and has poles of indefinitely high order.

Ferrers diagrams

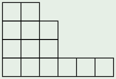
Generating function manipulations are okay, but if you have a theorem which claims that two sets have the same cardinality, a natural impulse is to ask for a rule to match the elements. Ferrers diagrams give an excellent way of visualizing partitions which can be manipulated in many ways.

Definition

The *Ferrers diagram* of $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a set of unit squares in the first quadrant justified to the axes, in which the column of squares with right edge over $x = i$ has highest edge at height λ_i.

Example

$\lambda = (4, 4, 3, 1, 1, 1) \vdash 14$

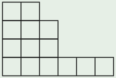


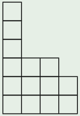
Ferrers diagrams

Before we reprove the theorem, we introduce a fundamental operation on Ferrers diagrams, namely *conjugation*: reflection across the main diagonal $x = y$. If a partition is fixed under conjugation, we say it is *self-conjugate*. We typically denote the conjugate of λ by λ'.

Example

$\lambda = (4, 4, 3, 1, 1, 1) \vdash 14$ $\quad$ $\lambda' = (6, 3, 3, 2)$





Ferrers diagrams

An easy result of conjugation is

Theorem

The number of partitions of n into distinct parts is equal to the number of partitions of n into consecutive parts (i.e., smallest part 1, and differences 0 or 1).

Proof.

If all the columns are of distinct lengths, the rows will increase in length by at most 1 at a time; vice versa, if the columns decrease so slowly, reading by rows will never give two of equal length. $\square$

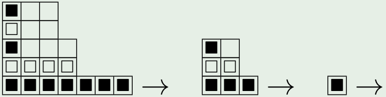
Conjugation (or its algebraic equivalent) is usually an ingredient of proofs for partition identities concerning difference conditions.

Ferrers diagram proof

Consider the Ferrers diagram of a partition into odd parts. Beginning with the largest part, we construct two parts out of each part of size at least 3, removing squares to produce new parts.

Split the largest part into $1 + \lfloor \frac{\lambda_1}{2} \rfloor + (\lambda'_1 - 1)$ and $\lfloor \frac{\lambda_1}{2} \rfloor + (\lambda'_2 - 1)$. Remove these and repeat.

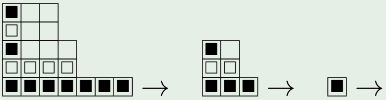
Example



(9,5,4,2,1)

Ferrers diagram proof

Example



(9,5,4,2,1)

Clearly this yields a partition of n into distinct parts, since each part is strictly smaller than the previous and the remaining partition keeps shrinking.

To reverse the process, add from the smallest parts. Notice that if we have an odd number of distinct parts, the smallest part must represent a string of 1s; if we have an even number of parts, 1 less than the difference between the last two must represent the excess of 1s extending beyond the previously added parts.

Ferrers diagram proof

This seems pretty laborious compared to our one-line generating function proof. But notice that our work has earned us a refined version of our original theorem:

Theorem

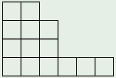
The number of partitions of n into odd parts **with no 1s** *is equal to the number of partitions of n into distinct parts* **where the difference between the two largest parts is exactly 1**.

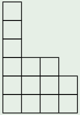
This is a frequent feature of "bijective" or "combinatorial" proofs, and hence the reason why they are commonly sought even when "analytic" proofs are known.

Exercise 2

Example

$\lambda = (4,4,3,1,1,1) \vdash 14 \quad \lambda' = (6,3,3,2)$





Sets of partitions which are closed under conjugation are equal in parity to the parity of their self-conjugate subset, since others can be paired off.

State some characteristics of a partition that are invariant under conjugation.

Ferrers diagram proof

Manipulating Ferrers diagrams can also prove generating function identities where the analytic proof is not so obvious.

The next theorem is called the *Pentagonal Number Theorem*, and provides a useful recurrence for the partition numbers.

Theorem

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}n(3n+1)} = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+\dots$$

Ferrers diagram proof

If we expand the product directly we have

$$\prod_{k=1}^{\infty}(1-q^k) = (1-q)(1-q^2)(1-q^3)\cdots = \sum_{n=0}^{\infty}(p_E(n) - p_O(n))\, q^n$$

where $p_E(n)$ and $p_O(n)$ are the numbers of partitions of n into an even and odd number of distinct parts respectively. So in turn, we would like to show that

Lemma

$p_E(n) - p_D(n) = (-1)^k$ if $n = \frac{1}{2}k(3k+1)$ for some $k \in \mathbb{Z}$, and 0 otherwise.

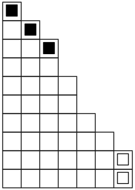
Ferrers diagram proof

When showing that the difference between two quantities is *usually* but *not always* zero, one strategy is to find a rule to match the two sets which occasionally breaks, and count the breaks.

We will construct an almost-involution on partitions into distinct parts which changes the parity of the number of parts, and is inapplicable to "excess" partitions of easily identifiable sizes.

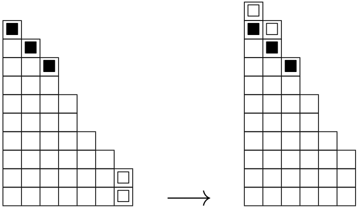
Ferrers diagram proof

Observe the Ferrers diagram of a partition into distinct parts:



Here we have $\lambda = (10, 9, 8, 6, 4, 3, 2)$. Notice that there are three sequences of consecutive parts: $(10, 9, 8)$, which we have marked in black, the singlet (6), and the last part, $(4, 3, 2)$.

Ferrers diagram proof



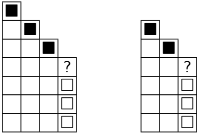
Because the smallest part is of size less than (or equal to) the length of the initial sequence of consecutive parts, we can move it atop those parts. Having done so, the new initial sequence is of length 2, and the new smallest part must be larger (distinct part sizes).

Ferrers diagram proof

So our basic map is, “if the smallest part is at most the length of the initial sequence of consecutive parts, move the smallest part to atop those parts; if the initial sequence is strictly shorter than the smallest part size, bring it down as a new part.”

Where does this map break?

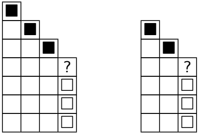
Ferrers diagram proof



If the smallest part is of length equal to the initial sequence, we would normally wish to place it atop the sequence, but if it is itself part of the initial sequence, there is not enough space.

Vice versa, if the initial sequence is 1 shorter than the smallest part, but the two again meet, then removing the tops of the sequence produces a part of equal size.

Ferrers diagram proof



So exceptional partitions into distinct parts are those that are of exactly the form

$$(2k-1, 2k-2, 2k-3, \ldots, k) \quad \text{or} \quad (2k-2, 2k-3, 2k-4, \ldots, k)$$

for $k \geq 1$. We leave as an exercise for the student that numbers of this form are precisely the pentagonal numbers $\frac{1}{2}n(3n+1)$ for $n \in \mathbb{Z}$.

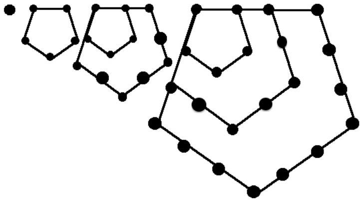
Pentagonal Numbers Exercise

No, seriously, students should do this exercise. Strategy:

1.) Observe that the pentagonal numbers from the formula can be written as the sums 1, $1 + (1 + 3)$, $1 + (1 + 3) + (1 + 3 + 3)$, etc., and 2, $2 + (2 + 3)$, $2 + (2 + 3) + (2 + 3 + 3)$, etc.
2.) Rewrite these uniquely as the sums described for various k.
3.) Observe that the number of summands correctly maps to the parity of n in $\frac{1}{2}n(3n + 1)$, i.e. if there are an odd number of summands, then n is odd.

Pentagonal Numbers

By the way, why are these called the pentagonal numbers? Here are the ones that arise for negative n:



For positive n, add another copy of any side of the outermost pentagon.

The Pentagonal Recurrence

Theorem

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}n(3n+1)} = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+\dots$$

We mentioned that the Pentagonal Number Theorem gives a recurrence for the partition numbers. Observe that

$$\frac{\prod_{k=1}^{\infty}(1-q^k)}{\prod_{k=1}^{\infty}(1-q^k)} = 1$$
$$= \left(1-q-q^2+q^5+q^7-\dots\right)\left(1+q+2q^2+p(3)q^3+p(4)q^4+\dots\right)$$

The Pentagonal Recurrence

So

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \ldots,$$

or, solving for $p(n)$,

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \ldots.$$

With decent memory space for list checks, this is a fairly fast way to compute the partition numbers.

Aside: Exact Formula and Asymptotics

There is a nearly closed formula; the derivation and its asymptotics are due to Hardy and Ramanujan, with improvements by Rademacher, and can be found in Andrews' textbook. For the curious, here it is:

Theorem

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[\frac{d}{dx} \frac{\sinh((\pi/k)(\frac{2}{3}(x-1/24))^{1/2}}{(x-1/24)^{1/2}} \right]_{x=n}$$

where $A_k(n)$ is a certain exponential sum involving $24k$th roots of unity, arising from the modular transformations of the η function $\eta(q) = q\prod_{j=1}^{\infty}(1-q^j)^{24}$.

Its main term is: $p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$.

Modified Pentagonal Recurrences

The Pentagonal Number Theorem gives us the reciprocal of the partition function, which is itself a useful tool. For later analysis, we will point out the following fact.

Consider partitions in which no part divisible by 5 appears. (These are known as the 5-*regular* partitions.) We can certainly write their generating function as

$$\sum_{n=0}^{\infty} b_5(n)q^n = \frac{\prod_{k=1}^{\infty}(1-q^{5k})}{\prod_{k=1}^{\infty}(1-q^k)}$$

$$= \left(1 - q^5 - q^{10} + q^{25} + q^{35} - \dots\right)\left(\sum_{n=0}^{\infty} p(n)q^n\right).$$

Modified Pentagonal Recurrences

By using the Pentagonal Number Theorem, we see that we can write $b_5(n)$ in terms of the partition numbers, namely

$$b_5(n) = p(n) - p(n-5) - p(n-10) + p(n-25) + p(n-35) - \dots .$$

We will discuss m-regular partitions later, and such recurrences will be useful for some theorems.