An Introduction to Partition Theory, Part II

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Let us introduce some useful notation. We denote the finite product

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \ldots (1 - aq^{n-1}), \quad (a; q)_0 = 1$$

and for the common case when $a = q$ we abbreviate

$$(q; q)_n = \prod_{j=1}^{n} (1 - q^j) =: (q)_n.$$

We also use the infinite product

$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n.$$

Another notation seen in combinatorics papers is the *q-factorial*,

$$[n]_q! = [1]_q[2]_q \dots [n]_q = \frac{(q; q)_n}{(1 - q)^n}$$
$$= (1 + q)(1 + q + q^2) \dots (1 + q + q^2 + \dots + q^{n-1}).$$

The reason for the name is clear – evaluation at $q = 1$ yields $n!$.

We say that such a polynomial is a *q-analogue* of the factorial. If we can show that a q-analogue of a number sequence is the generating function for a class of objects with some statistical weight, then we show that those objects are counted by the number sequence *and* that the set has some internal structure.

The classic example with the q-factorial is that $[n]_q!$ is the generating function for inversions in the permutations of length n:

$$[n]_q! = \sum_{\sigma \in S_n} q^{\operatorname{inv}\sigma}.$$

The statistic inv σ counts the number of inversions, pairs σ_i, σ_j in a permutation $\sigma = \sigma_1 \ldots \sigma_n$ such that $i < j$ but $\sigma_i > \sigma_j$. Building a permutation by inserting elements in order from 1 to n, it is clear that when inserting element k we can create from 0 to $k-1$ inversions simply by placing k a sufficient number of spaces from the end, and later insertions will not destroy any such inversion.

Thus this claim on the generating function for inversions refines the theorem that there are $n!$ permutations of length n.

We can write the generating function for partitions as $\frac{1}{(q)_\infty}$. Another identity is

$$\frac{1}{(q)_\infty} = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q)_n{}^2}.$$

To prove this latter identity, we mark the largest square we can find in the Ferrers diagram of a partition: this is known as the *Durfee square* of the partition. We then observe that every partition can be written as a Durfee square of some size n, and an ordered pair of partitions into parts of size at most n.

Define the *q-binomial coefficient*

$$\begin{bmatrix} n + m \\ m \end{bmatrix}_q = \frac{(q)_{n+m}}{(q)_n (q)_m}.$$

This is a *q*-analogue of the binomial coefficients, for when evaluated at $q = 1$, we obtain precisely the binomial coefficient $\binom{n+m}{m}$. We'll prove this by showing that the *q*-binomial coefficients are actually polynomials that are the generating functions for partitions into at most n parts of size at most m, i.e., they count partitions in the $n \times m$ box.

Consider the $n \times m$ box:
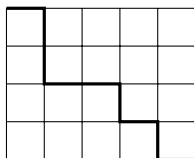
Partitions which fit in this box correspond (via their profiles) to lattice paths between opposite corners, and so presuming we can show that the q-binomials do indeed count them, their specialization immediately follows, since the $q = 1$ specialization is just the total number of such paths.

We begin by noting that $\begin{bmatrix} n \\ n \end{bmatrix}_q$ and $\begin{bmatrix} n \\ 0 \end{bmatrix}_q$ are both 1, simply because their numerators and denominators cancel.

Let the generating function for partitions in the $n \times m$ box be $f(n + m, m)$. The generating function for the single (empty) partition of 0 in an $n \times 0$ or $0 \times n$ box is just $q^0 = 1$. So the boundary conditions are the same for both functions. Now we will produce a doubly-indexed recurrence.
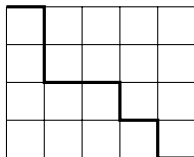
A partition in the box either has a part of largest possible size, or does not. This gives us a Pascal-like recurrence satisfied by the generating function for partitions in the box, which if satisfied by $\begin{bmatrix} n \\ m \end{bmatrix}_q$ proves their identity:

$$f(n + m, m) = q^m f(n - 1 + m, m) + f(n + m - 1, m - 1).$$

q-factorials and q-binomials

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q = q^m \begin{bmatrix} n-1+m \\ m \end{bmatrix}_q + \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}_q$$

Exercise 1: verify the algebra that confirms that the above identity holds. (Most factors in all three terms are common; deal with the remainder.)

This generating function is clearly a polynomial, since it is the generating function for a finite set of partitions. Here are a few:

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q = 1 + q + q^2 + \cdots + q^{n-1}$$

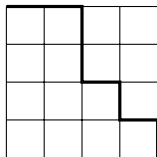since the $(n-1) \times 1$ box simply contains 1-row partitions of size up to $n-1$.

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

q-factorials and q-binomials

It is possible to construct a q-analogue of the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$:

$$\frac{1}{1+q+\cdots+q^n}\begin{bmatrix}2n\\n\end{bmatrix}_q.$$

This is the generating function for a certain statistic $maj(\pi)$ on *Dyck paths*, partitions which never go below the NW-SE diagonal:

We have two nice q-analogues of the binomial theorem (whole and general powers) which are very useful in proving q-series identities:

Theorem

$$\prod_{j=1}^{N}(1 + zq^j) = \sum_{m=0}^{N} q^{m(m+1)/2} \begin{bmatrix} N \\ m \end{bmatrix}_q z^m,$$

$$\prod_{j=1}^{N} \frac{1}{1 - zq^j} = \sum_{m=0}^{\infty} q^m \begin{bmatrix} N + m - 1 \\ m \end{bmatrix}_q z^m.$$

q-factorials and q-binomials

$$\prod_{j=1}^{N}(1 + zq^j) = \sum_{m=0}^{N} q^{m(m+1)/2} \begin{bmatrix} N \\ m \end{bmatrix}_q z^m$$

Observe that the coefficient on $q^n z^m$ on the LHS counts partitions of n into exactly m distinct parts at most N. From such a partition remove the largest possible triangle; there are at most m parts remaining, of size at most $N - m$, which are counted by $\begin{bmatrix} N \\ m \end{bmatrix}_q$.
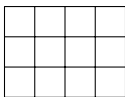
$N = 7, m = 5$:

$$\prod_{j=1}^{N} \frac{1}{1 - zq^j} = \sum_{m=0}^{\infty} q^m \begin{bmatrix} N + m - 1 \\ m \end{bmatrix}_q z^m.$$

Exercise: How do we prove this?

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix}_q = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6$$
$$+ 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12}$$

It is easy to notice that the q-binomials are *symmetric*: the degree of $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is $(n-m)m$, and the term of degree i has the same coefficient as the term of degree $(n-m)m - i$. Why?

We also have the binomial identity $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$. Why?

It is much less obvious that the coefficients are *unimodal*: they weakly increase, then weakly decrease, and only have one peak (sometimes extended, such as for $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$).

Questions of unimodality are of interest other classes of partitions: there are cases where we still want a combinatorial proof of something known analytically, and there are some cases where unimodality is open.

We will not prove this – it's rather difficult. Instead, let me point you to the quite readable papers where it was done, and discuss some context.

That the *q*-binomial coefficients are unimodal was first proved with abstract algebra, by J. J. Sylvester:

Sylvester, J. J., *Proof of the hitherto undemonstrated fundamental theorem of invariants*, Collect. Math. papers, Vol. 3, Chelsea, New York (1973), 117-126

We would rather have a proof that uses the properties of the partitions involved and breaks the *q*-binomial coefficient down into smaller sets that display the unimodality property. This was done much more recently, by Kathleen O'Hara:

O'Hara, K., *Unimodality of Gaussian Coefficients: A Constructive Proof*, Journal of Combinatorial Theory, Series A **53** 29-52 (1990)

O'Hara's proof decomposes the set of partitions in the $N \times M$ box into *chains*: subsets that are symmetric about the middle, including 1 each of partitions from some size i to size $NM - i$. If you do this, then no matter what the number or lengths of the chains required, their sum must certainly be a unimodal generating function.

Example

$$1 + q + \ q^2 + \ q^3 + \ q^4 + \ q^5 + \ q^6 + \ q^7 + q^8 + q^9$$
$$q^2 + \ q^3 + \ q^4 + \ q^5 + \ q^6 + q^7$$
$$q^3 + \ q^4 + \ q^5 + \ q^6$$
$$1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + q^8 + q^9$$

Unimodality

O'Hara's proof is quite readable on its own, but there is a nice
associated expositional paper by Doron Zeilberger which is almost
as well known:

Zeilberger, D., *Kathy OHaras constructive proof of the unimodality
of the Gaussian polynomials*, Amer. Math. Monthly **96** (1989), no.
7, 590-602

We will not go into O'Hara's paper in detail. Instead, I'd like to
place it in the context of the current state of knowledge and some
open questions about, and using, unimodality in partitions. To do
so, let's introduce some terms related to *partially ordered sets*.

Unimodality

A *partial order* on a set is a relation $<$ with which we can compare some, not necessarily all, members of a set. We require certain properties: irreflexivity ($x \not< x$), antisymmetricity ($x < y$ implies $y \not< x$), and transitivity. A set with a partial order is a *poset*.

If $x < y$ and there is no z such that $x < z < y$, we say y covers x. The *Hasse diagram* of a poset is a graph with the elements of the set as nodes, and an edge between x and y when x covers y.

A poset is *ranked* if it has subsets K_i such that elements in K_{i+1} only cover elements in K_i. The rank generating function of the poset is $\sum q^{rank(x)}$.

A *chain* is a sequence of elements $x_1 < x_2 < \dots$. An *interval* is a chain all of whose relations are covers.

With all these definitions, the partitions in the $N \times M$ box form a finite partially ordered set under the partial order of inclusion, or the Young order. We say $\pi_1 < \pi_2$ if π_1 has at most the same number of parts as π_2, and part j of π_1 is always at most the size of part j of π_2.

Covers occur when $\pi_1 \vdash n$ and $\pi_2 \vdash n+1$, so only one part is different. The ranking is obviously the weight of the partition.
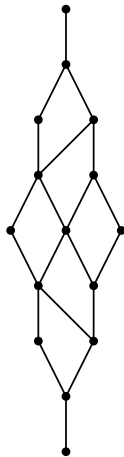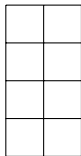
Let's look at a few $N \times M$ posets.
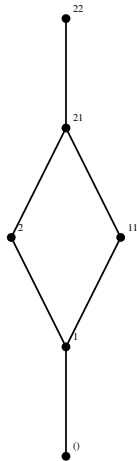
$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q ,$$

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q,$$

In these terms, the breakdown we asked for earlier to prove unimodality is called a *symmetric chain decomposition*. This is a decomposition of the poset into intervals symmetric about the middle rank. This is for the 3×3 box:
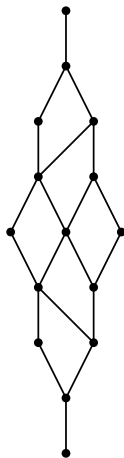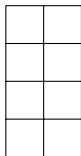
Example

$$1 + q + \ q^2 + \ q^3 + \ q^4 + \ q^5 + \ q^6 + \ q^7 + q^8 + q^9$$
$$q^2 + \ q^3 + \ q^4 + \ q^5 + \ q^6 + q^7$$
$$q^3 + \ q^4 + \ q^5 + \ q^6$$
$$1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + q^8 + q^9$$

Let's take a look at the 4×2 box again.

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q,$$
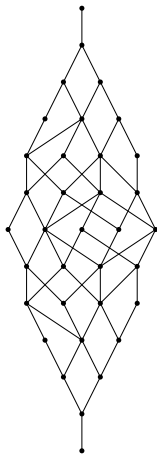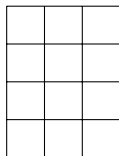
Can you find a symmetric chain decomposition?
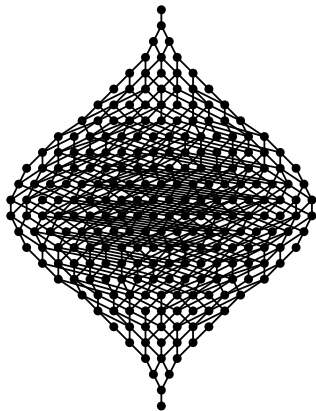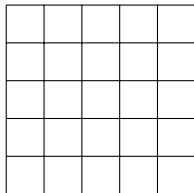
Alas, this gets rapidly harder as the number of parts increases:

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix}_q,$$

$$\begin{bmatrix} 10 \\ 5 \end{bmatrix}_q,$$

We could also use a weaker order, and just say that π_2 covers π_1 if $\pi_2 \vdash n + 1$ and $\pi_1 \vdash n$, so that every element on one rank covers every element on the one below. This gives us more chains but a less informative structure – however, it still proves unimodality, which knows nothing about the covering relations except rank.

In fact, this is what Kathleen O'Hara did. She said everything in one rank covers everything in the rank below, and used a theorem which constructs an SCD for the poset product of two posets which themselves have SCDs.

Therefore, although her work provided a symmetric chain decomposition for the $N \times M$ box, we were left with the open question:

Question

Can we find a symmetric chain decomposition of the $\begin{bmatrix} N \\ M \end{bmatrix}_q$ poset which respects the Young order?

Question

Can we find a symmetric chain decomposition of the $\begin{bmatrix} N \\ M \end{bmatrix}_q$ poset which respects the Young order?

This question *maaaaaay* have been recently solved. I still have to sit down with the paper.

However, we can ask the same question for another very natural set of partitions, and this is definitely unsolved.

Unimodality

Namely, consider partitions into distinct parts of size at most N: their generating function is

$$\prod_{i=1}^{N}(1 + q^i).$$

Exercise 2: Verify that this generating function is symmetric: multiply through by $q^{-N(N+1)/2}$, transform $q \to q^{-1}$, and show that the resulting polynomial is the same.

Exercise 3: Find a combinatorial bijection on the poset that realizes the symmetry. (How will you subtract parts?)

These polynomials are also known to be unimodal, but there is no known SCD, respecting Young's order or not.

Unimodality is useful because it can verify other properties. I'll close today's talk with the (First) Borwein Conjecture.

Conjecture

Define

$$\prod_{i=1}^{N}(1 - q^{3i-2})(1 - q^{3i-1}) := A(q^3) - qB(q^3) - q^2 C(q^3).$$

Then $A(q)$, $B(q)$, and $C(q)$ are all polynomials with nonnegative coefficients.

Equivalently, e.g., $A(q) = \sum_{n=0}^{\infty} \left(p_{3,N,e}(n) - p_{3,N,o}(n) \right) q^n$, where $p_{3,N,e}(n)$ is the number of partitions of $3n$ into an even number of parts not divisible by 3, of size at most $3N$, and likewise odd for $p_{3,N,o}$.

Borwein's conjecture says that for a number divisible by 3, there are at least as many partitions into an even number of such parts as there are into an odd number of such parts, and the reverse if the number is not divisible by 3.

Now, this question is easier if there is no restriction on size.

Theorem

(Andrews) Define

$$\prod_{i=1}^{\infty}(1 - q^{3i-2})(1 - q^{3i-1}) := A_{\infty}(q^3) - qB_{\infty}(q^3) - q^2 C_{\infty}(q^3).$$

Then $A_{\infty}(q)$, $B_{\infty}(q)$, and $C_{\infty}(q)$ are all power series with nonnegative coefficients.
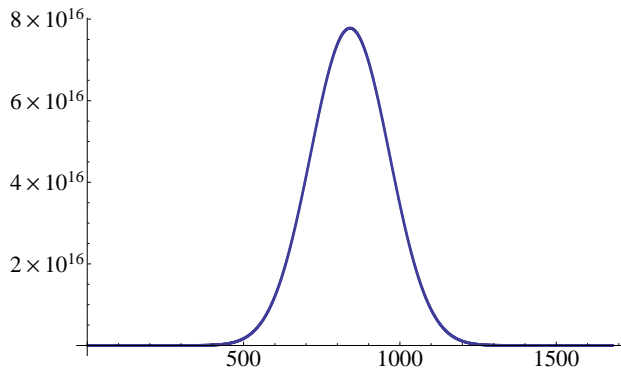
But if $n < 3N$, then there is essentially no size restriction, and the coefficients of $A(q)$ match those of $A_\infty(q)$. So the theorem holds in that range, and on the upper end of the range. (By the way, A is symmetric, but B and C are not; they are reflections of each other.)

Now, what do the coefficients of $A(q)$ look like?

For parts of size less than 120:

The polynomial certainly looks unimodal.

Since we know that it is nonnegative on the bottom, and on the top, then if we could show that $A(q)$ were unimodal, we would immediately have that all its coefficients were nonnegative.