# An Introduction to Partition Theory, Part III

Spring 2015

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# Partition congruences

If we observe a table of values of the partition numbers, striking regularities arise.

| | | | | | |
|---|---|---|---|---|---|
| $p(0) - p(4)$ | 1 | 1 | 2 | 3 | 5 |
| $p(5) - p(9)$ | 7 | 11 | 15 | 22 | 30 |
| $p(10) - p(14)$ | 42 | 56 | 77 | 101 | 135 |
| $p(15) - p(19)$ | 176 | 231 | 297 | 385 | 490 |
| $p(20) - p(24)$ | 627 | 792 | 1002 | 1255 | 1575 |

## Partition congruences

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The conjecture immediately arises that $p(5n+4) \equiv 0 \pmod 5$. With a little extra searching we can conjecture that $p(5^k n - 1) \equiv 0 \pmod{5^k}$, and that $p(7n+5) \equiv 0 \pmod 7$ and $p(11n+6) \equiv 0 \pmod{11}$, with corresponding results for powers of 7 and 11. Furthermore, these appear to be the only such progressions.

# Partition congruences

Ramanujan observed and conjectured these facts, which turn out to be *almost* all true. The sole exception is that $p(n)$ is not quite so neatly divisible by powers of 7. It turns out that $p(7^b n + \delta) \equiv 0 \pmod{7^{\lfloor (b+2)/2 \rfloor}}$ for the correct $\delta$, which is sufficient for $p(7n+5) \equiv 0 \pmod 7$ and $p(49+47)$ likewise, but not 343.

It is furthermore true that never again is $p(kn+j) \equiv 0 \pmod k$ for an entire residue class with any modulus other than $k = 5^a 7^{[(b+2)/2]} 11^c$, but this is *much* more difficult to prove.

# Partition congruences

The $p(5n+4)$ congruence is the most famous and the easiest to prove. The shortest proof I know of, which we will sketch, uses one of the two of the techniques common in partition theory papers: *dissections*.

In the second part of the talk, we will prove another congruence using another typical tool: *modular forms*.

The proof that I will sketch here largely comes from:

Hirschhorn, M.D., An identity of Ramanujan, and applications, in: *q-series from a contemporary perspective*, Contemporary Mathematics, Vol. 254, American Mathematical Society, 2000, 229-234.

It is available at
http://web.maths.unsw.edu.au/˜mikeh/webpapers/paper67.pdf .
Mike is a very clear writer who is good for a student to read.

The main tool that we will need for his argument is the *Jacobi Triple Product*, a powerful tool for manipulating $q$-series. The form we will use is

$$(a;q)_\infty(a^{-1}q;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n a^n q^{n(n-1)/2}.$$

The proof is not hard, about a page, and can be found in Andrews' *Theory of Partitions* or at http://mathworld.wolfram.com/JacobiTripleProduct.html .

The goal of the first part of the proof is to give an identity which allows us to display the subsequences of terms of $(q)_\infty$ in which the powers are 0 mod 5, 1 mod 5, 2 mod 5, et cetera.

Recall that the Pentagonal Number Theorem:

$$(q;q)_\infty = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \dots.$$

It turns out that only numbers 0, 1, and 2 mod 5 appear as powers here. Thus we will be able to write the *5-dissection*

$$(q;q)_\infty = A(q^5) + qB(q^5) + q^2C(q^5).$$

Our goal is to determine what these functions are.

We start with $(a;q)_\infty(a^2;q)_\infty(a^{-2}q;q)_\infty(a^{-1}q;q)_\infty(q)_\infty$.

Later, we will first make the substitution $q \to q^5$ and then $a \to q$, and then these five products will run over all the residue classes in $\prod_{k=1}^{\infty}(1-q^k)$.

We aim to show

$$(a;q)_\infty(a^2;q)_\infty(a^{-2}q;q)_\infty(a^{-1}q;q)_\infty(q)_\infty =$$

$$(q^5;q^5)_\infty\left(\frac{(a^5q;q^5)_\infty(a^{-5}q^4;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} - a\frac{(a^5q^2;q^5)_\infty(a^{-5}q^3;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\right.$$
$$\left.-a^2\frac{(a^5q^3;q^5)_\infty(a^{-5}q^2;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} + a^3\frac{(a^5q^4;q^5)_\infty(a^{-5}q;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\right)$$

Notice that each term is one residue class mod 5 in powers of $a$, and the $a^{5k+4}$ terms are 0.

If we multiply and divide by one factor of $(q;q)_\infty$, we get two triple products in the numerator:

$$\frac{\left((a;q)_\infty(a^{-1}q;q)_\infty(q;q)_\infty\right)\left((a^2;q)_\infty(a^{-2}q;q)_\infty(q;q)_\infty\right)}{(q;q)_\infty}.$$

We expand each triple using the Jacobi Triple Product, and gather terms involving $a$:

$$\frac{\left((a;q)_\infty(a^{-1}q;q)_\infty(q;q)_\infty\right)\left((a^2;q)_\infty(a^{-2}q;q)_\infty(q;q)_\infty\right)}{(q;q)_\infty}$$

$$=\frac{1}{(q;q)_\infty}\sum_{s=-\infty}^{\infty}(-1)^r a^r q^{(r^2-r)/2}\sum_{r=-\infty}^{\infty}(-1)^s a^{2s} q^{(s^2-s)/2}$$

$$=\sum_{n=-\infty}^{\infty}a^n c_n(q),$$

with

$$c_n(q)=\frac{1}{(q)_\infty}\sum_{r+2s=n}(-1)^{r+s}q^{(r^2-r+s^2-s)/2}.$$

Now we want to reverse the process and get the $c_n(q)$ as product generating functions in residue classes mod 5 for powers of $a$.
If we set $r = n - 2t$, $s = 2n + t$, then for $c_{5n}$ we get:

$$c_{5n}(q) = \frac{1}{(q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^{n+t} q^{((n-2t)^2-(n-2t)+(2n+t)^2-(2n+t))/2}$$

$$= (-1)^n q^{(5n^2-3n)/2} \frac{1}{(q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{(5t^2+t)/2}$$

$$= \frac{(-1)^n q^{(5n^2-3n)/2}}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

The last line in the previous frame follows from using the Jacobi Triple Product the other way.

Exercise: Show that

$$\sum_{t=-\infty}^{\infty}(-1)^t q^{(5t^2+t)/2} = (q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty.$$

We can find $c_{5n+1}(q)$, $c_{5n+2}(q)$, $c_{5n+3}(q)$, and $c_{5n+4}(q)$ similarly. We have $c_{5n+4}(q)$ ending up as

$$c_{5n+4}(q) = (-1)^{n+1} q^{(5n^2-5n+2)/2} \frac{1}{(q)_\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{(5t^2+5t)/2} = 0.$$

Why is this the 0 function?

Putting our sums together, we get

$$
\begin{aligned}
&(a, a^{-1}q, a^2, a^{-2}q, q; q)_\infty = \\
&\frac{1}{(q, q^4; q^5)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^{5n} q^{(5n^2-3n)/2} - \\
&\frac{1}{(q^2, q^3; q^5)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^{5n+1} q^{(5n^2-n)/2} \\
&- \frac{1}{(q^2, q^3; q^5)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^{5n+2} q^{(5n^2+n)/2} - \\
&\frac{1}{(q, q^4; q^5)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^{5n+3} q^{(5n^2+3nn)/2}.
\end{aligned}
$$

Finally, we use the Triple Product on each of the triangular sums once more to obtain the identity we sought:

$$(a;q)_\infty(a^2;q)_\infty(a^{-2}q;q)_\infty(a^{-1}q;q)_\infty(q)_\infty =$$

$$(q^5;q^5)_\infty\left(\frac{(a^5q;q^5)_\infty(a^{-5}q^4;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} - a\frac{(a^5q^2;q^5)_\infty(a^{-5}q^3;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\right.$$
$$\left.-a^2\frac{(a^5q^3;q^5)_\infty(a^{-5}q^2;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} + a^3\frac{(a^5q^4;q^5)_\infty(a^{-5}q;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\right)$$

Putting $q \to q^5$ and $a \to q$ in this evaluation, we obtain the dissection we desire:

$$(q)_\infty = (q^{25}; q^{25})_\infty \left( \frac{(q^{10}, q^{15}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty} - q - q^2 \frac{(q^5, q^{20}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty} \right)$$

where the $a^3$ becomes 0, because $(1; q)_\infty = (1-1)(1-q)\cdots = 0$.

For convenience, set $R(q) = \frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}$.

We are almost done. In

$$(q)_\infty = (q, q^2, q^3, q^4, q^5; q^5)_\infty = (q^{25}; q^{25})_\infty \left(R(q^5) - q - q^2 R^{-1}(q^5)\right),$$

substitute $q \to \zeta_5^j q$ where $\zeta_5$ is a complex 5th root of unity. (This is a common feature of dissection proofs). We obtain

$$(\zeta_5^j q, \zeta_5^{2j} q^2, \zeta_5^{3j} q^3, \zeta_5^{4j} q^4, q^5; q^5)_\infty = \\ (q^{25}; q^{25})_\infty \left(R(q^5) - \zeta_5^j q - \zeta_5^{2j} q^2 R^(q^5)^{-1}\right).$$

Now we multiply the results for all 5 powers, and make use of the facts that:

$$(1-\zeta_5 q)(1-\zeta_5^2 q)\dots(1-q) = 1-q^5,$$

and

$$(q,q^2,q^3,q^4;q^5)_\infty = \frac{(q;q)_\infty}{(q^5;q^5)_\infty}.$$

We obtain

$$\frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty} = (q^{25};q^{25})_\infty^5 \left(R(q^5)^5 - 11q^5 - q^{10}R(q^5)^{-5}\right).$$

Clear the left-hand side to 1 and divide through by $(q)_\infty$ on the left and $(q^{25};q^{25})_\infty \left(R(q^5) - q - q^2R^{-1}(q^5)\right)$ on the right: we get

$$\frac{1}{(q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \frac{R(q^5)^5 - 11q^5 - q^{10}R(q^5)^{-5}}{R(q^5) - q - q^2R^{-1}(q^5)}.$$

The denominator factors the numerator integrally:

$$\frac{1}{(q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \left(R(q^5)^4 + qR(q^5)^3 + 2q^2R(q^5)^2 + 3q^3R(q^5) + 5q^4 - 3q^5R(q^5)^{-1} + 2q^6R(q^5)^{-2} - q^7R(q^5)^{-3} + q^8R(q^5)^{-4}\right)$$

$$\frac{1}{(q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\left(R(q^5)^4 + qR(q^5)^3 + 2q^2R(q^5)^2 + 3q^3R(q^5)\right.$$
$$\left. + 5q^4 - 3q^5R(q^5)^{-1} + 2q^6R(q^5)^{-2} - q^7R(q^5)^{-3} + q^8R(q^5)^{-4}\right)$$

And now we have it, for the only powers $q^{5n+4}$ arise from the term

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = 5q^4\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}.$$

Thus, $p(5n+4)$ is always divisible by 5.

# Partition congruences

Congruences for higher powers of 5 need a deeper approach. We would require that the five functions $q\frac{(q^{25};q^{25})_\infty}{(q)_\infty}$ with $q \to \zeta_5^j q$ are the distinct roots of the quintic polynomial

$$u^5 - \frac{q^5(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6}(25u^4 + 25u^3 + 15u^2 + 5u + 1) = 0,$$

called the *modular equation of fifth order*. This would allow us to see the $p(5n+4)$ function as self-similar in useful ways.

The use of the theory of *modular forms* in proving partition identities will be the next part of the series, but before we discuss these, we digress to ask some natural questions about partition congruences.

# Partition congruences

First off, is there some more combinatorial way to show these identities? If we are claiming that some group of discrete objects is of a population divisible by 5, shouldn't we be able to divide them into 5 equal-sized groups? Or, at least, into a collection of groups all of size divisible by 5?

This is possible, but the proof that the construction works still relies heavily on the properties of modular forms.

# Partition congruences

Freeman Dyson discovered a simple statistic he called the *rank* of a partition: its largest part minus its number of parts. Amazingly, for $n \equiv 4 \pmod 5$, the partitions of $n$ with rank congruent to 0, 1, 2, 3, and 4 mod 5 form equinumerous classes.

| 0 mod 5 | 1 mod 5 | 2 mod 5 | 3 mod 5 | 4 mod 5 |
|---|---|---|---|---|
| 72 | 81 | 6111 | 9 | 711 |
| 51111 | 5211 | 531 | 621 | 63 |
| 4311 | 441 | 522 | 54 | 42111 |
| 4221 | 432 | 3211111 | 411111 | 3321 |
| 333 | 3111111 | 22221 | 33111 | 3222 |
| 2211111 | 222111 | 111111111 | 32211 | 21111111 |

# Partition congruences

The rank also shows the $p(7n+5)$ congruence, but not $p(11n+6)$. Dyson conjectured that some better statistic would show all three, and George Andrews and Frank Garvan found one, which they called the *crank*. It is defined as "the number of parts larger than the number of 1s minus the number of 1s if there are any 1s, else the largest part."

For instance, for $\lambda = 5+2+1+1$, the crank is $1-2=-1$, whereas for $5+4$ it is 5.

# Partition congruences

The rank and the crank explain the conguences, in a sense, but the proof relies heavily on the modularity of their generating functions. In other words, although they "display" the congruences, as proof techniques they are still somewhat lacking.

## Partition congruences

However, the crank is an extremely powerful tool.

The crank shows not only all three of Ramanujan's prime congruences, but all their powers. Furthermore, analysis of its modular properties showed the existence of congruences for *all* numbers not divisible by 2 or 3, although the progressions have moduli which are large multiples of the modulus of the progression.

# Partition congruences

There are two obvious holes here.

First, is there some way to produce a more combinatorial explanation of the congruences where the proof does not rely on the modularity properties of the generating function? Possibly not, but this is still an area of some interest. Time permitting, I will discuss some explorations in this direction where development might be suitable for grad students next session.

# Partition congruences

The second is a major and motivating question in partition theory: what about the residue of $p(n)$ mod 2 and mod 3?

**Wide open!**

# Partition congruences

Here's what we know and don't know:

Radu showed that there exist *no* arithmetic progressions $An + B$ for which the partition function is constant mod 2 or mod 3. This was recent and hard: proving *non*congruences is very different.

The obvious conjecture is that parity and tertiarity are equally distributed in the limit, i.e. the number of odd partition numbers $p(n)$ for $n < X$ should tend to $\frac{1}{2}X$.

We know that this number is at least $\frac{\sqrt{X}}{\ln X}$.

It is not known whether the density of odd $p(n)$ is positive. Even getting the number to $\sqrt{X}$ would be an achievement.