An Introduction to Partition Theory, Part IV

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Modular techniques

We described last session the surprisingly difficult open question of whether the partition numbers $p(n)$ are equally often even and odd, i.e. whether it is true for each i that

$$\lim_{N\to\infty} \frac{1}{N}\#\{p(n) \equiv i \pmod 2 \,|\, n \leq N\} = \frac{1}{2}$$

and likewise for modulus 3, whether

$$\lim_{N\to\infty} \frac{1}{N}\#\{p(n) \equiv i \pmod 3 \,|\, n \leq N\} = \frac{1}{3}$$

Very little is known about these conjectures. But perhaps surprisingly, a relatively small change to the set of partitions we're looking at can result in some very strong congruences mod 2 or 3, and powers of these.

Modular techniques

Recall that we are sometimes interested in partitions into parts not divisible by some m. The more usual word for these is m-regular partitions. They have a simple generating function:

$$B_m(q) = \sum_{n=0}^{\infty} b_m(n)q^n = \frac{(q^m; q^m)_\infty}{(q; q)_\infty}.$$

Modular techniques

Our first theorem is

Theorem

$$B_9(q) = \frac{(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^2 (q^6,q^{30};q^{36})_\infty} + q\frac{(q^{12};q^{24})_\infty^2 (q^{36};q^{36})_\infty}{(q^4;q^4)_\infty (q^4;q^8)_\infty^6} + 3q^3\frac{(q^{24};q^{24})_\infty^2 (q^{36};q^{36})_\infty}{(q^4;q^4)_\infty^3 (q^4;q^8)_\infty^2}$$

with its immediate corollary

Theorem

$b_9(4n+3) \equiv 0 \pmod 3$.

Modular techniques

In order to prove this identity, we will rewrite the theorem in terms of the η function. We let $q = e^{2\pi iz}$, and define

$$\eta(z) = q^{1/24}(q;q)_\infty.$$

Hence $\eta(kz) = q^{k/24}(q^k;q^k)_\infty$.

As an aside,

$$\tau(z) = \eta^{24}(z) = q(q;q)_\infty^{24}$$

is known as the Ramanujan τ function. *Lehmer's conjecture* is that the coefficients of this function are all nonzero. We know that almost all are, and the conjecture holds up to 10^{23}.

Modular techniques

In our desired identity

$$B_9(q) = \frac{(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^2(q^6,q^{30};q^{36})_\infty} + q\frac{(q^{12};q^{24})_\infty^2(q^{36};q^{36})_\infty}{(q^4;q^4)_\infty(q^4;q^8)_\infty^6} + 3q^3\frac{(q^{24};q^{24})_\infty^2(q^{36};q^{36})_\infty}{(q^4;q^4)_\infty^3(q^4;q^8)_\infty^2}$$

notice that not all of our factors are η functions.
In order to rewrite our theorem, we need two observations:

$$\frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \quad \text{and} \quad \frac{1}{(q,q^5;q^6)_\infty} = \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty}{(q;q)_\infty(q^6;q^6)_\infty}.$$

Modular techniques

In terms of η products, our desired identity becomes

$$q^{-1/3}\frac{\eta(9z)}{\eta(z)} = q^{-1/3}\frac{\eta(12z)^2\eta(18z)\eta(12z)}{\eta(2z)^2\eta(6z)\eta(36z)} + q^{-1/3}\frac{\eta(8z)^6\eta(36z)\eta(12z)^2}{\eta(4z)^7\eta(24z)^2} + 3q^{-1/3}\frac{\eta(8z)^2\eta(36z)\eta(24z)^2}{\eta(4z)^5}.$$

Multiplying through by a factor of $q^{1/3}\eta(4z)^4$, we obtain

$$\frac{\eta(9z)\eta(4z)^4}{\eta(z)} = \frac{\eta(12z)^2\eta(18z)\eta(12z)\eta(4z)^4}{\eta(2z)^2\eta(6z)\eta(36z)} + \frac{\eta(8z)^6\eta(36z)\eta(12z)^2}{\eta(4z)^3\eta(24z)^2} + 3\frac{\eta(8z)^2\eta(36z)\eta(24z)^2}{\eta(4z)}.$$

Modular techniques

A natural question would be "I get the $q^{1/3}$, but why the $\eta(4z)^4$?"

In order to explain this we introduce the idea of *modular forms*, a special class of functions in a complex variable z. If $f(z)$ is a function holomorphic (complex differentiable) on $\mathbb{H}$ without poles on $\mathbb{R} \bigcup \{i\infty\}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all elements of $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), c \equiv 0 \pmod{N}\}$ with N minimal, then f is a modular form of weight k and level N.

Modular techniques

One may show that the property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

holds for $\Gamma_0(N)$ by checking that it holds for the generators; if $N = 1$, these are translation, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \to z+1$, and inversion, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \to -1/z$.

Thus, we can loosely say that modular forms are functions that behave well under translation and inversion.

Modular techniques

Products and quotients of η functions are often modular forms. We can guarantee that they will be under circumstances outlined in the theorem of Gordon, Hughes and Newman:

Theorem

Let $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ *with* $r_\delta \in \mathbb{Z}$. *If*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \text{ and } \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ *is a modular form of weight* $k = \frac{1}{2}\sum r_\delta$ *and level* N.

Modular techniques

The weight and level are important because a theorem of Sturm tells us that we can check that two modular forms are equal by checking a finite number of their coefficients, up to a bound dependent on their weight and level:

Theorem

Let $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=0}^{\infty} b(n)q^n$ both be modular forms of weight k and level N. If

$$p \mid (a(n) - b(n)) \quad \forall \quad 0 \leq n \leq \frac{k}{12} N \cdot \prod_{d \text{ prime}: d \mid N} \left(1 + \frac{1}{d}\right),$$

then $f(z) \equiv g(z) \bmod p$.

Modular techniques

Why do we only need to check a finite number of coefficients to ensure congruence (or equality, if two forms are congruent for all primes) for an infinite series?

Because *the set of modular forms of a given level and weight form a finite-dimensional vector space over* $\mathbb{C}$.

Modular techniques

Back to our desired identity:

$$\frac{\eta(9z)\eta(4z)^4}{\eta(z)} = \frac{\eta(12z)^2\eta(18z)\eta(12z)\eta(4z)^4}{\eta(2z)^2\eta(6z)\eta(36z)} + \frac{\eta(8z)^6\eta(36z)\eta(12z)^2}{\eta(4z)^3\eta(24z)^2} + 3\frac{\eta(8z)^2\eta(36z)\eta(24z)^2}{\eta(4z)}.$$

Notice that if we take the lcm of all the δ that appear in a term, we can also take any multiple of this for our N. That allows us to match N for all of our terms. (If the weight didn't match we'd be out of luck.) With GHN we can thus see that each term is a modular form of weight 2 and level dividing $N = 216$.

Modular techniques

The Sturm bound for modular forms of weight 2 and level (dividing) 216 is

$$\frac{2}{12} 216 \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) = 72.$$

We use *Mathematica* to expand both sides of the identity as q-series, up to the q^{72} term. We find that they are equal, and so every term of both sides is congruent modulo every prime. The only way that can happen is if they are exactly equal, and thus the identity (and its corollary) is established.

Modular techniques

With modular techniques, the work is essentially reduced to finding a conjectured congruence, constructing a related modular form, and verifying it. It's no surprise that there is a fairly substantial literature of congruences for any number of m-regular partitions.

m	$An + B$	$\equiv 0$ mod...	m	$An + B$	$\equiv 0$ mod...
4	(9,4)	4	7	(9,8)	3
4	(9,7)	12	10	(9,3)	3
5	(4,2)	2	13	(9,7)	3
5	(20,13)	2	19	(81,33)	3
9	(4,3)	3	22	(27,16)	3
16	(81,77)	2	25	(27,17)	3

There are still a couple of things missing, however.

We just showed that $b_9(4n+3) \equiv 0 \pmod{3}$. Is there a combinatorial explanation? Can you construct three equal-sized classes, or at least classes of size divisible by 3, to justify the claim? In short, is there a crank? (The actual crank does not work.)

Modular techniques

Is there a general theorem? With so many congruences for m-regular partitions running around, for so many m, one would think that we could glean some insight into saying that $b_m(An+B) \equiv 0 \pmod{N}$ for some infinite class of m and/or arithmetic progressions. However, while I have heard one claim of a forthcoming paper to this effect, I have not seen anything in black and white.

Modular techniques

Now we'll use modular forms in a different way. Instead of proving equality, we'll just prove that all coefficients of a q-series are congruent to 0 modulo a prime.

This is handy when it is hard to conjecture or work with the proper form of a generating function. Here, we will be looking at the function $\nu_2(n)$, which counts the number of partitions of n into parts of exactly 2 different sizes. For instance, $\nu_2(6) = 6$, counting

$5+1$, $4+2$, $4+1+1$, $3+1+1+1$, $2+2+1+1$, $2+1+1+1+1$

but not 6, 3+3, 3+2+1, 2+2+2, or 1+1+1+1+1+1.

Modular techniques

Student Exercise: Justify the claim that

$$\sum_{n=0}^{\infty} \nu_1(n)q^n = \sum_{k=1}^{\infty} \frac{1}{1-q^k} = \sum_{n=0}^{\infty} d(n)q^n,$$

where $d(n)$ is the number of divisors of n.

Modular techniques

Theorem

(MacMahon) $\nu_2(n) = \frac{1}{2}\left(\sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n)\right)$

Proof sketch: Consider a "two-sided Ferrers diagram" built by separating n into k and $n-k$, both nonzero, and creating two rectangles, one of area k and the other of area $n-k$. The number of such diagrams is $\sum_{k=1}^{n-1} d(k)d(n-k)$. You will take half the number of these after removing pairs in which the rectangles are the same height (overcounting slightly).

Modular techniques

$\nu_2(n) = \frac{1}{2}\left(\sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n)\right)$

Using this, we will show:

Theorem

$\nu_2(16n+14) \equiv 0 \pmod{4}$.

Surprisingly, the proof comes down to counting representations of numbers by sums of squares, and using a modular form congruence to show that there are an even number of desirable pairs.

I thank Jeremy Rouse of Wake Forest University for showing me the modular forms argument that finishes off this technique.

Modular techniques

$$\nu_2(n) = \frac{1}{2}\left(\sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n)\right)$$

First observe that $\sigma_1(16n+14) \equiv 0 \pmod 8$, because $16n + 14 = 2(8n+7)$, and for $8n+7$, one of two things happens: either it has a factor which is 7 mod 8 which appears to an odd power, or it has factors 3 and 5 mod 8 which both appear to odd powers. (If you're not sure, work this out!)

In the former case, let p be the prime $8k+7$. Then every factor has either a 2 or not, and p an odd number of times or an even. If even, call it x. Then factors can be grouped as x, $2x$, $(8k+7)x$, and $2 * (8k+7)x$, and the total of these is $(24k+24)x$. The argument is similar for the other case.

Modular techniques

So we can toss out $\frac{1}{2}\sigma_1(16n+14)$. That means we have reduced the problem to showing, for $m = 16n+14$,

$$\nu_2(m) = \frac{1}{2}\left(\sum_{k=1}^{m-1} d(k)d(m-k) + d(m)\right) \equiv 0 \pmod{4}.$$

Next, observe that $d(m) \equiv d\left(\frac{m}{2}\right)^2 \pmod{8}$, and both are divisible by 4.

Divisors of m can be grouped as x, $2x$, px, and $2px$, and the total is either 0 or 4 mod 8. Half of this is either 0 or 2 mod 4, but when we square this we obtain either 0 or 4 mod 8 respectively.

Modular techniques

Thus, you can toss out the $d(m)$ and the term $k = m/2$. Finally, since $k \neq m - k$ for $k \neq m/2$, we get each pair $(k, m - k)$ twice, and so it suffices to show that either half of the remaining sum is 0 mod 4:

$$\sum_{k=1}^{\frac{m-2}{2}} d(k)d(m-k) \equiv 0 \pmod 4.$$

$$\sum_{k=1}^{\frac{m-2}{2}} d(k)d(m-k) \equiv 0 \pmod 4.$$

Now, notice that $d(k)$ is only odd when k is a square. Otherwise, factors come in pairs. But, $16n + 14 \equiv 6 \pmod 8$, so it can never be the sum of two squares (squares are always 0, 1, or 4 mod 8).

So either k or $m - k$ is a nonsquare. If both are, the term is 0 mod 4, so we reduce to cases where one of them is a square.

$$\sum_{k=1}^{\frac{m-2}{2}} d(k)d(m-k) \equiv 0 \pmod{4}.$$

Furthermore, if either $d(k)$ or $d(m-k)$ is 0 mod 4, we can throw that term out, so the nonsquare term will have to be 2 mod 4: $d(k)$ or $d(m-k)$ must be $d(py^2)$ where p is some prime that divides y to an even power, say $2j$. Then $d(py^2) = (4j+2)d(y^2) \equiv 2 \pmod{4}$.

Thus we reduce to the following claim:

Theorem

The number of representations of $m = 16n + 14$ in the form $x^2 + py^2$, where p is a prime that divides y to an even power, is even.

If this is the case, then we will have an even number of terms (one for each k the smaller of x^2 and py^2) contributing 2 mod 4, and everything else contributing 0 mod 4, and the total will be divisible by 4.

Modular techniques

If $x^2 + py^2 \equiv 6 \pmod 8$, it is either the case that x is twice an odd number, $p = 2$, and y is odd, or that x is odd, y is odd, and $p \equiv 5 \pmod 8$.

We now have two very important congruences:

$$F(q) := \sum_{n=0}^{\infty} \sigma_1(2n+1)q^{2n+1} \equiv \sum_{n=1}^{\infty} q^{(2n+1)^2} \pmod 2$$

and

$$G(q) := \frac{1}{2}\sum_{n=0}^{\infty} \sigma_1(8n+5)q^{8n+5} \equiv \sum_{\substack{p\equiv 5 \pmod 8 \\ y\geq 1, 2|s_p(y)}} q^{py^2} \pmod 2.$$

Modular techniques

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That $F(q)$ is a modular form of weight 2 and level 4 can be proved by hand, using the generators for $\Gamma_0(4)$. $G(q)$ is one of the terms of its dissection, and it can be shown that such a term is also a form of the same weight and higher level (in this case, $N = 256$). This is rather tricky and so we won't dwell on it.

Modular techniques

$$F(q) := \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1} \equiv \sum_{n=1}^{\infty} q^{(2n+1)^2} \pmod 2$$

$$G(q) := \frac{1}{2} \sum_{n=0}^{\infty} \sigma_1(8n+5) q^{8n+5} \equiv \sum_{\substack{p \equiv 5 \pmod 8 \\ y \geq 1, 2 | s_p(y)}} q^{py^2} \pmod 2$$

The proofs of the congruences rely on several facts about $\sigma_1(n)$: $\sigma_1(c^2) \equiv 1 \pmod 2$ if c is odd; $\sigma_1(n)$ is multiplicative; and $\sigma_1(p^n) = 1 + p + p^2 + \cdots + p^n$.

Strongly Recommended Student Exercise: prove the above congruences using the given facts.

Modular techniques

The generating function for the number of representations of $8n+6$ of the form x^2+py^2 is congruent mod 2 to

$$H(q) = F(q^4)F(q^2) + F(q)G(q).$$

Why? Recall that $F(q) \equiv \sum_{n=1}^{\infty} q^{(2n+1)^2} \pmod 2$, so $F(q^4) \equiv \sum_{n=1}^{\infty} q^{4(2n+1)^2} \pmod 2$, the generating function for the squares of twice the odd numbers. Then $F(q^2) \equiv \sum_{n=1}^{\infty} q^{2(2n+1)^2} \pmod 2$ counts $2y^2$ with y odd. Likewise, $F(q)G(q)$ counts x^2+py^2 where $p \equiv 5 \pmod 8$.

$$H(q) = F(q^4)F(q^2) + F(q)G(q)$$

We've done two new things here, so we need two more facts about modular forms:

If $f(q)$ is a modular form of weight k and level N, then $f(q^\ell)$ is a modular form of weight k and level ℓN, and

if $g(q)$ is also a modular form of weight ℓ and level N, then $f(q)g(q)$ is a modular form of weight $k + \ell$ and level N.

Putting these together, we obtain that $H(q)$ is a modular form of weight 4 and level 256.

Modular techniques

The Sturm bound for a modular form of weight 4 and level 256 is $\frac{4}{12}256\left(1+\frac{1}{2}\right)=128$, and we calculate that all coefficients of $H(q)$ up to the Sturm bound are even. (In fact, all of those that are not in the progression $8n+6$ are 0.) Thus, they all are.

$H(q)$ counts the number of representations of $8n+6$, including $16n+14$, that give terms 2 mod 4 in

$$\sum_{k=1}^{\frac{m-2}{2}} d(k)d(m-k).$$

Since there are an even number of such terms, the full sum is 0 mod 4. Hence $\nu_2(16n+14) \equiv 0 \pmod{4}$, as claimed. $\square$

Modular techniques

The functions $\nu_k(n)$ are not much explored, but they seem quite interesting. It's easy to show that there are quite a lot of progressions where $\nu_2(An+B) \equiv 0 \pmod{2}$. There are a number of different open questions that might be tackled:

1. Predict $An+B$ for which $\nu_2(An+B) \equiv 0 \pmod{4}$.
2. $\nu_3(An+B) \equiv 0 \pmod{2}$ occurs occasionally. Find more.
3. $\nu_3(An+b) \equiv 0 \pmod{2}$ seems to occur only if $\nu_2(An+B) \equiv 0 \pmod{4}$ (not only if). Is that true?
4. If so, what is a sufficient condition for the linkage?
5. Does a congruence mod anything other than 2 or 4 ever occur? I have not found one yet.
6. Does ν_4 or higher ever have congruences? I suspect not.

I hope you've enjoyed this series and learned a little bit about what partition theorists like to study, and how we study it. I'm happy to discuss proof techniques and research questions in detail with anyone who would like to ask. For giving me four sessions of the Combinatorics Seminar,

THANK YOU!