A Spectral Graph Sparsification Approach to Scalable Vectorless Power Grid Integrity Verification

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ABSTRACT
Vectorless integrity verification is becoming increasingly critical to robust design of nanoscale power delivery networks (PDNs). To dramatically improve efficiency and capability of vectorless integrity verifications, this paper introduces a scalable multilevel integrity verification framework by leveraging a hierarchy of almost linear-sized spectral power grid sparsifiers that can well retain effective resistances between nodes, as well as a recent graph-theoretic algebraic multigrid (AMG) algorithmic framework. As a result, vectorless integrity verification solution obtained on coarse level problems can effectively help find the solution of the original problem. Extensive experimental results show that the proposed vectorless verification framework can always efficiently and accurately obtain worst-case scenarios in even very large power grid designs.

CCS CONCEPTS
• Hardware → Very large scale integration design; • Computing methodologies → Modeling and simulation;

GENERAL TERMS
Performance, Algorithms, Verification

KEYWORDS
Vectorless verification, spectral graph theory, graph sparsification, algebraic multigrid

1 INTRODUCTION
Power grid integrity verification is key to designing reliable power delivery networks (PDNs) for nanoscale integrated circuits. To avoid the prohibitively high cost of traditional vector-based verification methods that rely on running numerous circuit simulations using over-pessimistic current distributions, a series of vectorless power grid verification techniques has been investigated in the past decade [2–4, 6–8, 11, 14, 17]. Recent research has made good progress in reducing the power grid verification costs by using novel sparse approximate inverse (SPAI) technique [6], efficient dual algorithm [17], and node elimination [7]. Despite these significant improvements, the overall power grid verification cost is still extremely high, especially for large-scale verification tasks.

To significantly improve the verification efficiency, scalable multilevel vectorless verification methods based on geometric multigrid (GMG) operations and PDE-constrained optimization framework have been recently proposed [3, 4, 9] to tackle large scale flip-chip power grid integrity verification problems. However, such methods usually require the underlying power grid designs to have relatively regular structures so that GMG operations can be performed effectively, which can become a major limitation when applied to nanoscale PDN designs where regular power grid structures are rare to find.

This paper introduces a scalable multilevel power grid verification framework that leverages a recent graph-theoretic algebraic multigrid (AMG) algorithmic framework [10] as well as a hierarchy of almost linear-sized power grid sparsifiers. The proposed vectorless verification approach gains insights from prior multilevel PDE-constrained optimization methods [9], circuit adjoint sensitivity analysis and spectral graph theory [16]. The original multilevel optimization method assumes that once given a hierarchy of model problems ordered from the finest to the coarsest levels, the optimization solution can be incrementally improved on coarser to coarsest level problems, while the coarser level optimization solution can effectively facilitate finding the optimal solution for the original problem.

By taking advantage of recent spectral graph sparsification techniques [5, 19] that can well retain the effective resistances of original grid, we are able to preserve important spectral properties of power grid even on coarse levels, which is key to efficient and accurate multilevel vectorless power grid verifications: worst-case integrity verification solutions obtained on coarse level problems will be very likely to match the result of the original problem. Consequently, it becomes highly scalable to tackle a series of global and local vectorless integrity verifications from the coarsest to finest levels. Additionally, the proposed approach enables to effectively trade off solution quality and verification efficiency in a rather straightforward manner. Our extensive experimental results demonstrate that the proposed approach allows to compute adjoint sensitivities in much faster ways (in almost linear time) and solve much smaller linear programs (LPs) compared to the original vectorless verification problems.

The rest of this paper is organized as follows. Section 2 provides a brief introduction to prior research work on vectorless power grid verification and spectral graph sparsification. Section 3 describes technical components of the proposed verification framework. Section 4 demonstrates extensive experimental results of
vectorless verification for a variety of real-world, large-scale power grid designs, which is followed by the conclusion of this work in Section 5.

2 BACKGROUND

2.1 Vectorless Integrity Verification

The DC analysis of an $n$-node power grid can be formulated into following equation by utilizing nodal analysis[3, 6]:

$$G x = b,$$  \hspace{1cm} (1)

where $G$ is a conductance matrix representing all the interconnected resistors in the power grid, $x$ is an $n \times 1$ node voltage vector, and $b$ is the right-hand side current vector. The transient analysis of the RC power grid is similar to DC analysis, which can be formulated as follows:

$$C \dot{x}(t) + G x(t) = b(t).$$  \hspace{1cm} (2)

Vectorless power grid voltage or current integrity verification aims to identify the maximum voltage drops or current densities under linear current constraints [4, 6], where current constraints are introduced to capture current loading variations and correlations in given chip design. There are both local constraints and global constraints in a typical vectorless verification problem: the local constraints set the upper bounds for each current source while global constraints set the upper bounds for blocks of individual current sources. For example, vectorless DC voltage integrity verification is equivalent to solving the following LP problem [3]:

$$\text{maximize : } v_i = x_i^T G^{-1} b, \text{ for } i = 1, \ldots, n,$$

subject to the following current constraints:

- Local Constraints : $b^L \leq b \leq b^U$
- Global Constraints : $0 \leq Q b \leq b_g$,

where $x_i$ is an elementary unit vector with the $i$-th entry being 1 and others being zeros; the inverse of $G$ only contains nonnegative sensitivity values since $G$ is an $L$-matrix; the $b^L$ and $b^U$ denote lower bounds and upper bounds of current sources, while $Q$ is an $m \times n$ matrix representing $m$ global current constraints.

2.2 Multilevel Vectorless Verification

Recent work in [3, 4] proposed a scalable geometric multigrid (GMG) based vectorless verification method that can efficiently handle very large power grid designs leveraging a multilevel PDE-constrained optimization framework, which is briefly described as follows:

1. Build coarse level power grids as well as constraints using GMG-like operations.
2. Perform adjoint sensitivity analysis to identify global and local critical regions.
3. Perform vectorless verifications starting from the coarsest level and map worst case current vectors to finer levels, which is similar to the GMG prolongation operations.
4. Set up new LP problems for local critical regions and perform solution refinements to improve solution quality.

Although the above multilevel method has a very good scalability, it requires the GMG hierarchies and inter-grid operators to be defined in advance, which may not be feasible for general power grid designs that do not have regular geometric structures.

2.3 Spectral Graph Sparsification

Consider a weighted, undirected graph $G = (V, E, \omega)$, where $V$ is the vertex set of the graph, $E$ is the edge set of the graph, and $\omega$ is a function that assigns every edge a positive weight. The Laplacian matrix of graph $G$ can be defined as follows:

$$L_G(p, q) = \begin{cases} -w(p, q) & \text{if } (p, q) \in E \\ \sum_{(p, t) \in E} w(p, t) & \text{if } (p = q) \\ 0 & \text{otherwise}. \end{cases}$$  \hspace{1cm} (4)

The Laplacian matrix can be considered as the conductance matrix for a resistor network. For any real vector $x \in \mathbb{R}^V$, the Laplacian quadratic form of graph $G$ can be defined as:

$$x^T L_G x = \sum_{(p, q) \in E} w_{p, q} (x(p) - x(q))^2.$$  \hspace{1cm} (5)

Graph sparsification tries to find an ultra sparse subgraph (graph sparsifier) so that the subgraph can preserve important features of the original graph. For example, the cut-based graph sparsifier preserves the value of the cuts in a graph, whereas spectral sparsifier preserves eigenvalues and eigenvectors of the original graph Laplacian. Recent research work shows that spectral sparsifier is always a stronger notion than the cut sparsifier [16].

Graphs $G$ and $P$ are $\sigma$-spectrally similar if the following is satisfied for all real vectors $x \in \mathbb{R}^V$ according to [1]:

$$\frac{x^T L_P x}{\sigma} \leq x^T L_G x \leq \sigma x^T L_P x.$$  \hspace{1cm} (6)

It also has been shown that spectrally similar graphs always have similar effective resistances or distances between vertices, which consequently allows much faster graph-based algorithms to be developed [16]. To construct spectrally-similar subgraphs, effective resistance based edge sampling scheme and a spectral perturbation based approach have been previously proposed [5, 15]. Although both methods have nearly-linear time complexities, the spectral perturbation based approach is considered more practically efficient for handling large-scale graph problems [5].

3 VECTORLESS VERIFICATION WITH SPECTRAL SPARSIFICATION

3.1 Motivation of Our Approach

To overcome the limitation of previous GMG-based multilevel vectorless verification methods, we introduce a new multilevel approach leveraging AMG-like inter-grid operations as well as spectral graph sparsification. The proposed method is a more general approach, since it directly works on a given power grid conductance matrix, and does not require any geometric information of power grids during the multilevel hierarchy setup phase and vectorless verification phase. More specifically, our approach is based on a latest AMG research work, Lean Algebraic Multigrid (LAMG), that takes into consideration of spectral properties of graph Laplacians [10], which allows to construct the multilevel coarse grid problems without loss of much accuracy. However, the coarse level problems obtained by LAMG usually become increasingly denser, leading to dramatically increased computational cost for adjoint sensitivity analysis. To address this challenging issue, an efficient
spectral graph sparsification method [5] is applied to not only maintain a series of tree-like power grid sparsifiers but also approximately preserve effective resistances between nodes when setting up coarse level problems.

It should be noted that preserving effective resistances (relative distances) between nodes on coarse-level grids is ultimately important for multilevel verification problems since it is equivalent to preserving adjoint sensitivities that will be eventually used for setting up LPs. As a result, the adjoint sensitivities for each LP solve can be computed in nearly linear time without loss of much accuracy, while the number of variables involved in the LPs during vectorless verifications can also be dramatically reduced due to the aggressive aggregations of current sources (variables) on sparsified coarse-level grids.

Another motivation behind the proposed approach is that in many optimization methods (like interior-point method for solving LPs), as long as the search direction (e.g., effective resistances or adjoint sensitivities) is guaranteed to be good, the final solution (e.g., worst case current vector) quality should also be good. We also emphasize that although the proposed approach is suitable for both vectorless voltage and current integrity verification problems, we will only focus on voltage integrity problems for clarity purpose.

3.2 Overview of the Proposed Approach

Fig. 1 describes the overall flow of the proposed method for vectorless power grid verification. It includes two phases: (a) a setup phase for creating multilevel power grid sparsifiers and (b) a multilevel vectorless verification phase for identifying worst case integrity scenarios. Phase (a) includes the following key steps for setting up everything:

1. Step 1. Perform spectral sparsification for the original power grid to reduce network complexity without changing the grid size.
2. Step 2. Perform nodal elimination and aggregation on the sparsified grid to recursively create multilevel power grid sparsifiers.
3. Step 3. Factorize power grid sparsifiers at different levels for adjoint sensitivity computations.
4. Step 4. Define AMG-like restriction (prolongation) operators for constraints (solution) mapping operations at different levels according to the nodal elimination and aggregation schemes obtained at Step 2.

Phase (b) includes the following key steps for vectorless verification of a specific node or grid region:

1. Step 1. Compute adjoint sensitivities using the matrix factors for the power grid sparsifier at each level.
2. Step 2. Identify a global critical region on the coarsest level and local critical regions on finer levels.
3. Step 3. Start vectorless verification by setting up LPs at the coarsest level to obtain the initial solution vector.
4. Step 4. Recursively interpolate solution vectors to a finer level and perform local solution refinements at each level until reaching the finest level.

3.3 Multilevel Power Grid Sparsifiers

We only consider the following two grid problem that can be generalized for creating other coarse grid levels. To generate a coarse-level (sparsified) grid problem based on a fine-level (sparsified) grid problem, the following node elimination and aggregation steps have been proposed to reserve spectral (topological or global) properties of graph Laplacians based on the LAMG algorithm [10]: First, a nodal elimination procedure is performed to eliminate low-degree nodes, which will be very efficient since power grid sparsifiers have only tree-like structures with most node degrees being 1 or 2; Next, a node aggregation procedure is performed for aggregating strongly connected nodes together according to a nodal affinity metric $c_{uv}$ that is defined as follows for nodes $u$ and $v$:

$$c_{uv} = \frac{\| (x_u, x_v) \|^2}{(x_u, x_u)(x_v, x_v)}$$

where $x_u = (x_u^{(1)} \ldots x_u^{(K)})$ is formed by applying a few Gauss-Seidel relaxations using $K$ test vectors to the problem $L_p x = 0$. The nodal affinity $c_{uv}$ can be shown to reflect the distance or strength of connection between nodes $u$ and $v$. For example, a greater $c_{uv}$ value indicates a stronger connection between nodes $u$ and $v$ [10]. Consequently, nodes with large affinity values can be aggregated together to form the nodes on the coarse-level grid.

3.4 Mapping Operations for Current Constraints

The current constraints in vectorless verification include both local and global constraints. The local current constraints $b_h^U$ (upper bound) and $b_h^L$ (lower bound) for the current source vector $b$ at level $h$ can be directly mapped to coarser level $H$ to obtain the upper bound $b_H^U$ and lower bound $b_H^L$ using AMG’s restriction operator $V_H^H$ obtained during the aforementioned multilevel coarse grid construction phase, which is shown as follows:

$$Upperbound: \quad b_H^U = V_H^H b_h^U, \quad Lowerbound: \quad b_H^L = V_H^H b_h^L.$$ 

Mapping global current constraints can be quite similar, by keeping coarse grid block current bounds to be the sum of block current bounds on the fine grid. As a result, local and global current constraints can be obtained for each multilevel problem, which will be later used during the multilevel vectorless verification procedure.
3.5 Solution Mapping and Refinement

To reduce the verification cost on the coarsest level, we first define the global critical region \( C_{glb} \) based on the adjoint sensitivity threshold [3]. For example, \( C_{glb} \) will only include the most important current sources (with top sensitivity values) that can significantly impact voltage drops or current densities. Given a sensitivity threshold \( s_{th} \) or normalized sensitivity threshold \( \epsilon_{glb} = s_{th}/s_{max} \), where \( s_{max} \) is the maximum adjoint sensitivity of all current sources, \( C_{glb} \) can be obtained by finding all current sources whose adjoint sensitivity values are greater than \( s_{th} \) or \( \epsilon_{glb} \) and subsequently including them into \( C_{glb} \) for setting up LPs. As a result, the global verification on the coarsest level can be formulated as:

\[
\text{maximize : } v_{wst} = \sum_{\forall b_i \in C_{glb}} s_i b_i \tag{8}
\]

subject to local and global current constraints:

\[
b^L \leq b \leq b^U, \quad 0 \leq Q b \leq b_y. \tag{9}
\]

Once the solution \( b^H_{wst} \) on the coarsest level is obtained, it is mapped back to the finer level using the AMG prolongation operator \( v^H_{wst} \) determined during the setup phase:

\[
\tilde{v}^H_{wst} = v^H_{wst} b^H_{wst}. \tag{10}
\]

Since directly mapping the solution of the coarsest level problem to a finer level may lead to increased solution error, a local solution refinement procedure at the finer level becomes essential as suggested in [3]. To this end, we propose to incrementally improve the solution quality on the finer grid by setting up a new LP for a much smaller local critical region and combining the updated LP solution with interpolated coarse-level solution to gain good efficiency and accuracy. The key steps in the proposed solution mapping and refinement procedure for a voltage integrity verification problem have been shown below:

(1) Set the normalized sensitivity threshold \( \epsilon_{loc} = \beta \epsilon_{glb} \) with the scaling factor \( \beta > 1 \) to obtain a much smaller local critical region;

(2) Set up a new LP problem for the local critical region \( C_{loc} \) to obtain the solution vector \( \tilde{v}^H_{wst} \);

(3) For the current sources (variables) that belong to \( C_{loc} \), update their solution with \( \tilde{v}^H_{loc} \); for the sources (variables) that belong to \( C_{glb} \) but not \( C_{loc} \), reuse the interpolated solution \( \tilde{v}^H_{wst} \);

(4) Compute the refined solution (worst case voltage drop) by:

\[
v_{wst} = \tilde{v}_{wst} + \epsilon_{glb} \cdot s_{max} b^h - b^h_{wst/1}.
\]

A detailed multilevel power grid voltage integrity verification algorithm has been described in Algorithm 1.

4 EXPERIMENTAL RESULTS

Extensive experiments have been conducted to validate the proposed vectorless verification method that is implemented in MATLAB and C++. The LP problems are solved by the LP solver published in [13] and all results are measured using a single CPU core of a computing platform running 64-bit RHEW 6.0 with 2.67GHz 12-core CPU and 48GB DRAM memory. The test cases include industrial power grid designs with different sizes up to 9 million nodes [12, 18].

Algorithm 1 AMG-based Power Grid Vectorless Voltage Verification with Spectral Sparsification

\begin{enumerate}
  \item Extract spectral sparsifier for the original power grid.
  \item Multilevel coarse grid construction:
    \begin{enumerate}
      \item Construct all hierarchy levels from finest to coarsest level;
      \item Set up local and global current constraints \( b^L, b^U \) and \( Q \) for each level with AMG mapping operators \( V^H_b \) and \( V^H_h \);
    \end{enumerate}
  \item Perform the matrix factorizations for each coarse-level grid.
  \item Perform global verification at the coarsest level \( K \):
    \begin{enumerate}
      \item Identify global critical region \( C^K_{glb} \) for a given normalized sensitivity threshold \( \epsilon_k \);
      \item Set up LP solver to get the worst case current vector \( b^H_{wst} \);  
    \end{enumerate}
  \item Perform solution mapping and refinement on finer to finest levels:
    \begin{enumerate}
      \item \( k \leftarrow K \)
      \item while \( k > 1 \) do
        \begin{enumerate}
          \item Interpolate solution vector to finer level by: \( b^{k-1}_{wst} = V^H_k b^H_{wst} \)
          \item Set sensitivity threshold \( \epsilon_{k-1} = \beta \epsilon_k \) and identify \( C^k_{loc} \);
          \item Setup a new LP for \( C^k_{loc} \) to obtain current vector \( b^H_{loc} \);
          \item Combine the latest LP and interpolated solutions to form \( b^{k-1}_{wst} \);
        \end{enumerate}
      \end{enumerate}
    \end{enumerate}
  \item Return the verification results (current source vector).
\end{enumerate}

![Figure 2: Relative error of vectorless verification w/ sparsified grids.](image)

4.1 Result of Solution Quality

As we mentioned in the previous sections, the spectral graph sparsification method can well preserve the effective resistances of the original power grid, which will guarantee a good solution quality during vectorless verifications. Fig.2 shows very satisfactory results for the vectorless verifications with the sparsified power grids (single level).

4.2 Result of Runtime Efficiency

Adjoint sensitivity calculation for vectorless power grid verification includes two parts: a matrix factorization phase, and a linear equation solving phase using matrix factors. For example, for a given power grid conductance matrix \( G \), to find the voltage sensitivity at a specific node \( i \) with respect to all current sources, the
Table 1: Results of the proposed vectorless power grid integrity verification method.

<table>
<thead>
<tr>
<th>Power Grid Specs.</th>
<th>Single Level</th>
<th>Multilevel w/o Sparsiﬁcation</th>
<th>Multilevel w/ Sparsiﬁcation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CKT</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ibmpg3</td>
<td>0.85M</td>
<td>90K</td>
<td>2</td>
</tr>
<tr>
<td>ibmpg4</td>
<td>1.0M</td>
<td>100K</td>
<td>2</td>
</tr>
<tr>
<td>ibmpg6</td>
<td>1.7M</td>
<td>170K</td>
<td>2</td>
</tr>
<tr>
<td>ibmpg7</td>
<td>1.5M</td>
<td>150K</td>
<td>2</td>
</tr>
<tr>
<td>thupg1</td>
<td>5.0M</td>
<td>500K</td>
<td>3</td>
</tr>
<tr>
<td>thupg2</td>
<td>9.0M</td>
<td>900K</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2: Runtime results of the proposed method.

<table>
<thead>
<tr>
<th>CKT</th>
<th>Single Level</th>
<th>Original Multilevel</th>
</tr>
</thead>
<tbody>
<tr>
<td>ibmpg3</td>
<td>12.4X</td>
<td>47.8X</td>
</tr>
<tr>
<td>ibmpg4</td>
<td>6.0X</td>
<td>26.2X</td>
</tr>
<tr>
<td>ibmpg6</td>
<td>3.0X</td>
<td>28.4X</td>
</tr>
<tr>
<td>ibmpg7</td>
<td>6.2X</td>
<td>30.5X</td>
</tr>
<tr>
<td>thupg1</td>
<td>8.3X</td>
<td>43.8X</td>
</tr>
<tr>
<td>thupg2</td>
<td>20.4X</td>
<td>93.8X</td>
</tr>
</tbody>
</table>

$t_{resolve}$ denotes the runtime for linear equation solving. It shows dramatic speedups in sensitivity calculations with the proposed spectral graph sparsiﬁcation procedure.

4.3 Comprehensive Results of the Proposed Method

Comprehensive veriﬁcation results using different approaches are shown in Table 1, with speedup numbers shown in Table 2. N, C, L are the numbers of grid nodes, current sources, and hierarchy levels, respectively; “Single Level”, “Multilevel w/o Spariﬁcation”, and “Multilevel w/ Spariﬁcation” denote the veriﬁcation methods using single level (direct), multilevel method w/o and w/ sparsiﬁcation methods, respectively; $t_{chol}$, $t_{sol}$, and $t_{lp}$ denote the runtime for Cholesky factorizations, adjoint sensitivity calculation using matrix factors and total LP solve including all levels, respectively; $Err$ denotes the relative error of maximum voltage drop compared to single level method and $\kappa$ denotes the relative condition number.

For all test cases, it is observed that matrix factorization and sensitivity calculation procedures in “Multilevel w/ Spariﬁcation” method are the fastest among all three methods, while “Multilevel w/o Spariﬁcation” is always the slowest due to the fast growing matrix densities at coarse levels. The LP solution time $t_{lp}$ with “Multilevel w/ Spariﬁcation” is also the smallest since the number of variables and constraints in vectorless veriﬁcation can be more signiﬁcantly reduced on the sparsiﬁed grids than the original grid.

It is observed that solving LPs using the proposed method is over 2X faster than using the “Multilevel w/o Spariﬁcation” method, and over 20X faster than using the “Single Level” method, showing that the proposed method can play very important roles in reducing overall computational cost during vectorless veriﬁcations, especially for large power grid designs. Fig. 4 shows the nearly-linear runtime scalability of the proposed method, where both the LP solve time and adjoint sensitivity calculation time have been demonstrated.

Figure 3: Sensitivity computation time for the original and sparsiﬁed grids.

right-hand-side (RHS) vector is set to be $b = [0, \ldots, 1, \ldots, 0, \ldots, 0]$ with only the $i$-th entry being 1 and other entries being 0. By solving the linear system of equations based on matrix factors, the solution $x$ can be obtained and used as the sensitivity vector for setting up the following LPs. The matrix factors only need to be computed once for each level and will be reused many times during the vectorless veriﬁcation process. Since the conductance matrix of a sparsiﬁed power grid can be factorized and solved in almost linear time, adjoint sensitivity computations based upon sparsiﬁed grid can be much more efﬁcient than the original grid problem. As shown in Fig.3, the runtime results of sensitivity calculation for the original and sparsiﬁed power grids have been illustrated, where $t_{chol}$ denotes the runtime for Cholesky matrix factorization,
5 CONCLUSION
We present a scalable multilevel vectorless power grid integrity verification approach that does not rely on any geometric information of power grid designs. Our approach takes advantages of recent spectral graph sparsification, graph-theoretic AMG algorithm research and multilevel PDE-constrained optimization methods to achieve good efficiency and accuracy for vectorless power grid verification problems. The proposed method leverages a hierarchy of almost linear-sized spectral power grid sparsifiers that can well retain the original effective resistance, as well as efficient graph-theoretic AMG operations. As a result, worst-case integrity verification solution obtained on coarse level problems will effectively facilitate computing the verification solution for the original problem. Extensive experimental results show that the proposed vectorless verification framework can always efficiently obtain worst-case scenarios in large power grid designs without loss of accuracy.

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