

EMTP Program

Perhaps a skewed view of history & development. Note also ATP.

Basic Features:

- V source
- I source
- XFMRs
- R, L, C
- T-Lines
- Synch Gens
- Ind Motors.

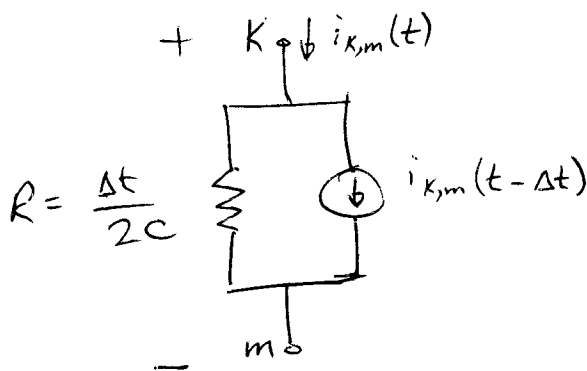
Implementation:

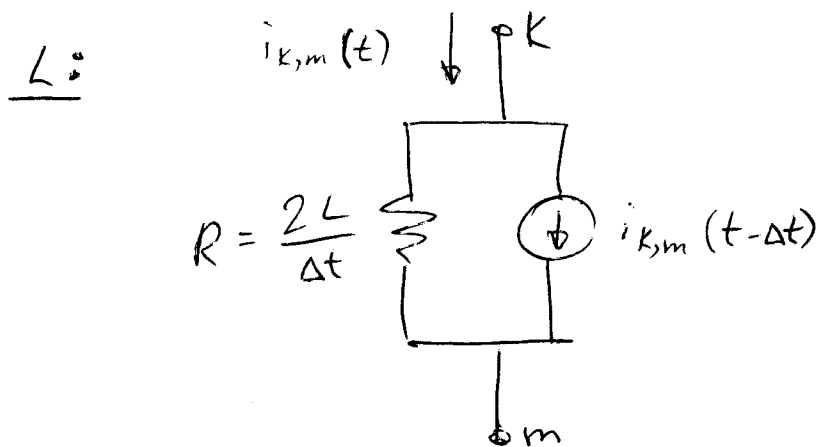
$[Y_{bus}]$ is augmented as per discussion in EE580. (Give copy of handout).

R:



C:





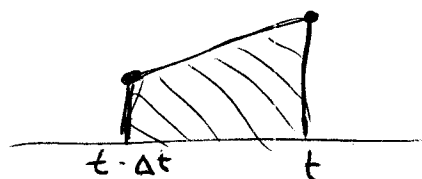
Ex:

$$v_k - v_m = L \frac{di_{k,m}}{dt} \Rightarrow$$

$$di_{k,m} \approx i_{k,m}(t) - i_{k,m}(t-\Delta t)$$

$$i_{k,m}(t) \approx i_{k,m}(t-\Delta t) + \frac{1}{L} \int_{t-\Delta t}^t (v_k - v_m) dt$$

Δi



By trapezoidal rule (assume small Δt)

~~$$i_{k,m}(t) = \frac{\Delta t}{2L} [v_k(t-\Delta t) - v_m(t-\Delta t)] + i_{k,m}(t-\Delta t)$$~~

$$i_{k,m}(t) = i_{k,m}(t-\Delta t) + \frac{\Delta t}{2L} [v_k(t) - v_m(t) + v_k(t-\Delta t) - v_m(t-\Delta t)]$$

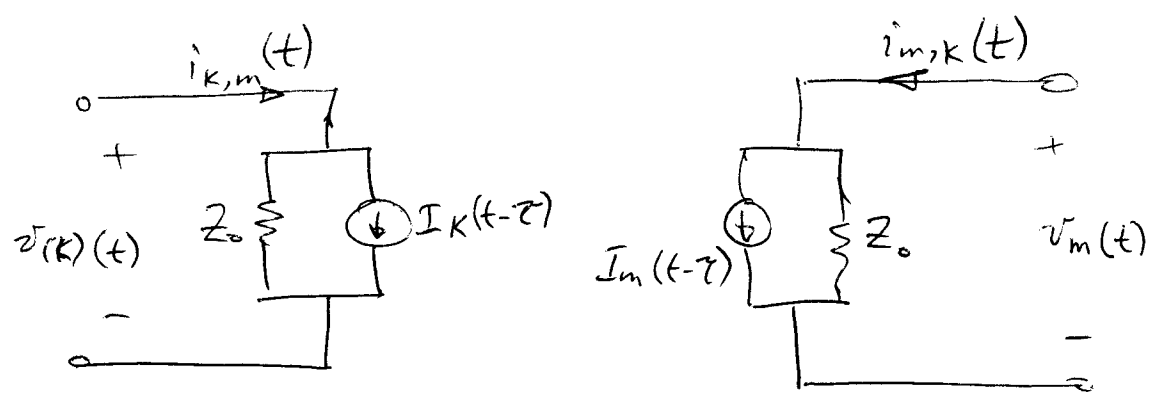
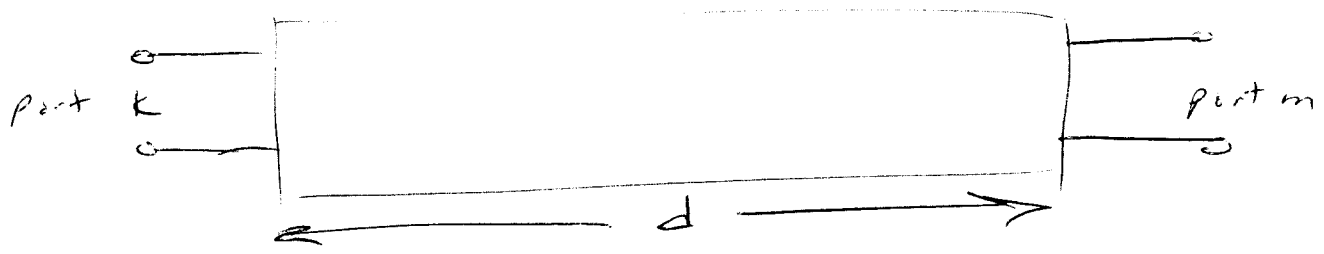
Separating (t) & $(t-\Delta t)$ terms,

$$i_{k,m}(t) = \underbrace{\left(\frac{\Delta t}{2L} [v_k(t) - v_m(t)] \right)}_{\text{conductance}} + i_{k,m}(t-\Delta t) + \underbrace{\frac{\Delta t}{2L} [v_k(t-\Delta t) - v_m(t-\Delta t)]}_{\text{past history}}$$

Conductance that gives current at (t) for present voltage across it.

$i_{k,m}$ from past history (actually the summation of all past current). Start out with $i(0)$.

Transmission Line:



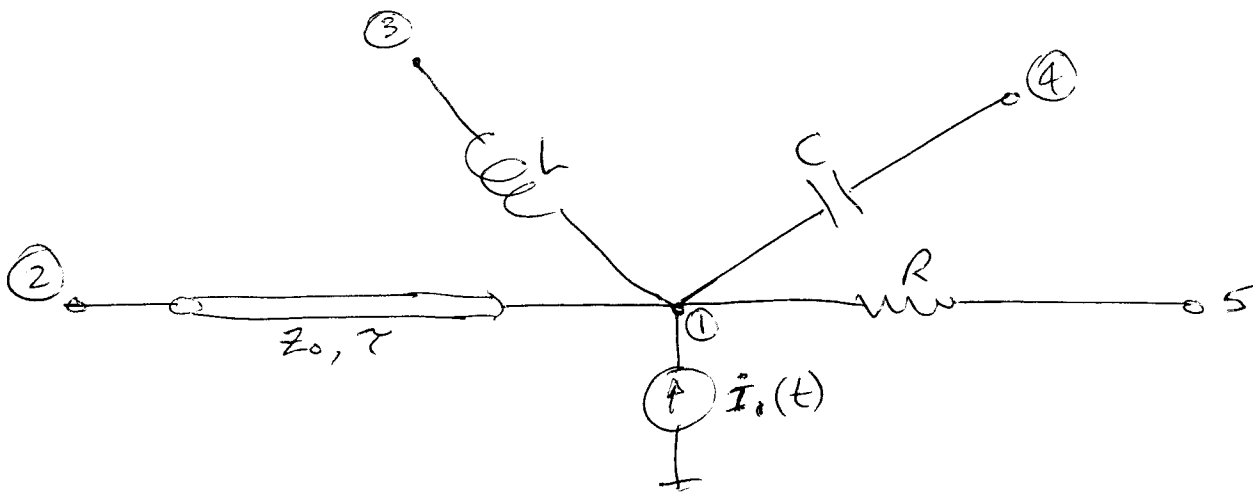
$$\tau = \text{travel time} = \frac{l}{v} = \frac{d}{\sqrt{LC}}$$

Lossless lines in air:
 Note $v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{LC}} = c = 3 \times 10^8 \text{ m/s}$

~~2~~ $2 < \epsilon_r < 5+$ for cable
 $\mu_r > 1$ for windings in machine or xfmr,

Air insulated line: 1000 ft/ms $\tau = 5.37 \text{ us/mile}$
 $= 3.34 \text{ ns/Km}$

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$$i_1(t) = i_{12} + i_{13} + i_{14} + i_{15}$$

- All elements are composed/implemented as Rs and current sources.

$$\left(\frac{1}{Z_0} + \frac{\Delta t}{2L} + \frac{2C}{\Delta t} + \frac{1}{R} \right) v_1(t) - \frac{\Delta t}{2L} v_3(t) = \frac{2C}{\Delta t} v_4(t) - \frac{1}{R} v_5(t)$$

$$= i_1(t) - I_{12}(t-\tau) - I_{13}(t-\Delta t) - I_{14}(t-\Delta t) - I_{15}(t-\Delta t)$$

$$[G][v(t)] = [i(t)] - [I]$$

↑ past history current terms

Some node voltages are established/known if v-source is present.

$$\begin{bmatrix} G_{AA} & G_{AB} \\ G_{BA} & G_{BB} \end{bmatrix} \begin{bmatrix} v_A(t) \\ v_B(t) \end{bmatrix} = \begin{bmatrix} i_A(t) \\ i_B(t) \end{bmatrix} - \begin{bmatrix} I_A \\ I_B \end{bmatrix}$$

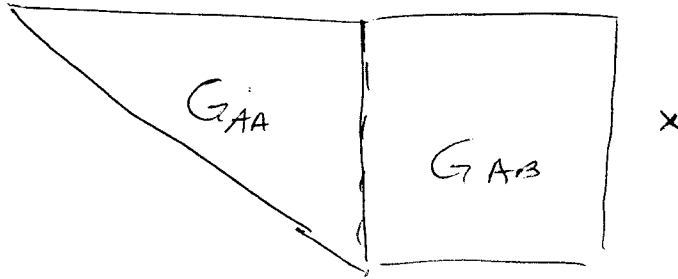
So equations are partitioned.

$$\therefore [G_{AA}][i_A(t)] = [I_{TOTAL}] - [G_{AB}][i_B(t)]$$

$$\uparrow$$

$$= [i_A(t)][I_A]$$

↑ note typo in book.



- Sparse matrices,
- LU factorization.

Alternative Numerical Integration Methods for use with the Companion Circuit Method for Electric Power Circuit Solutions

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Abstract

Circuit transient analysis packages such as SPICE and EMTP are very widely used, but detailed information on the internal structure and error characteristics for these software packages is lacking. Both SPICE and EMTP rely on numerical integration to approximate the transient response of circuit elements. The numerical integration methods used in the packages are not necessarily the most accurate approximations to the actual response of a circuit. The study of these numerical integration methods and their error characteristics is the focus of this paper. Comparisons are made between several methods in terms of accuracy and stability as well as algorithm and time complexity involved with implementing the integration methodologies.

1. Introduction

This paper concerns numerical integration techniques used in the resistive companion circuit method for the calculation of electrical transients. The motivation for this work came originally from the Electromagnetic Transients Program (EMTP) [1,2] which is widely used computer software designed for the analysis of electric power networks. The essence of EMTP software is the use of resistive equivalent circuits ("companion circuits") which model general RLC networks. Once the network has been reduced to a resistive companion, numerical integration is used to calculate bus voltages and injection currents at discrete time intervals. The main objective of this paper is to study alternative integration methods in this application.

The key to understanding the companion circuit methodology of transient analysis is to understand the numerical integration scheme that is employed. Also, one must comprehend how to formulate the companion circuits. This is the motivation for an exposition of alternative numerical integration methodologies and techniques for the formulation of the nodal admittance matrix. There are many numerical integration techniques which could be used in the application of companion circuits. This paper will present six methods which fall into the two categories of: first order approximations and second order approximations. The following methods will be presented: forward Euler, backward Euler, trapezoidal, parabolic, Simpson's, and

Gear's second order methods [3,4]. The first three methods are first order approximations. The integral of a first order method is solved by calculating the area under a linear approximation of the function. The last three methods are second order approximations. These methods approximate an integral by solving the area under a parabolic (or second order) region. Numerical integration relies on the successive summation of a number of discrete time slices of the function, each separated by a time step (h). Formulas for these techniques appear in Table 1.

Table 1 Comparative Properties of Alternative Numerical Integration Methods

Method	Basis of Method	Integration Formula	Order of Approximation	Self Starting?
1	Forward Euler	$x_{n+1} = x_n + hx'_n$	1	yes
2	Backward Euler	$x_{n+1} = x_n + hx'_{n+1}$	1	yes
3	Trapezoidal Rule	$x_{n+1} = x_n + \frac{h}{2}[x'_n + x'_{n+1}]$	1	yes
4	Simpson's Rule	$x_{n+1} = x_n + \frac{h}{3}[x'_{n-1} + 4x'_n + x'_{n+1}]$	2	no
5	Parabolic	$x_{n+1} = x_n + \frac{h}{12}[-x'_{n-1} + 8x'_n + 5x'_{n+1}]$	2	no
6	Gear's 2nd Order	$x_{n+1} = \frac{4}{3}x_n - \frac{1}{3}x_{n-1} + \frac{2h}{3}x'_{n+1}$	2	no

2. Error analysis

Numerical methods generally result in errors associated with each integration step (the exception occurring when the approximation is of a simple rational number which corresponds to a function of equal or lesser order). The main error components are due to round-off and truncation of the mathematical approximation to the integral. Round-off errors are the errors associated with using finite digital computer word lengths. These types of errors are machine as well as language dependent. Round-off errors are present any time the exact number desired requires more digits to be represented than the computer uses. The second type of error associated with numerical integration methods is the truncation, or theoretical, error. Truncation error is independent of the machine and language used; it is the error associated with the algorithm employed.

Each numerical integration method approximates integrals in various ways. The truncation error is dependent on the way (and to what order) this approximation is made. Table 2 contains a summary of the integration methodologies and the anticipated errors associated with each. These errors are derived from the Taylor series expansion of an integral approximation. Each method results in a different truncation of the Taylor series, thus resulting in different anticipated errors. A more detailed view of these errors is found in Chua and Lin [3]. (It should be noted that the value of the time step (h) is important in determination of the error characteristics of an integration method.)

Table 2 Error Approximations of Methods 1-6

Method	Error Approximation *	Order of Approximation
Forward Euler	$\frac{h^2 x^{(2)}(\tau)}{2}$	$O(h^2)$
Backward Euler	$-\frac{h^2 x^{(2)}(\tau)}{2}$	$O(h^2)$
Trapezoidal	$-\frac{h^3 x^{(2)}(\tau)}{12}$	$O(h^3)$
Simpson's Rule	$-\frac{h^5 x^{(5)}(\tau)}{5!}$	$O(h^5)$
Parabolic	$-\frac{3h^4 x^{(4)}(\tau)}{160}$	$O(h^4)$
Gear's 2nd Order	$-\frac{2h^3 x^{(3)}(\tau)}{9}$	$O(h^3)$

* $t_n < \tau < t_{n+1}$

3. Stability analysis

As an algorithm passes through each time step, it is desired that the total error (summation of the local truncation errors) decays with time. An algorithm which exhibits this behavior is said to be numerically stable. Stability is a function of the equation to be solved, the time step of integration, and the method of numerical integration employed. The methods shown above can be studied for their regions of stability in an attempt to further realize the merits of each method. Following the analysis of stability in Chua and Lin [3], the study of stability will be performed using an equation of the form,

$$x' = -\lambda x \quad x(0) = 1$$

which is a first order system with time constant of $1/\lambda$. The analytic solution of this is,

$$x(t) = x_0 e^{-\lambda t} \quad t \geq 0$$

where x_0 is the initial condition. The study of stability can be performed by placing the λx value into the integral formulas shown in Table 1. The forms of solution for the first three methods are as follows,

Forward Euler:

$$x_{n+1} = (1 + h\lambda)x_n.$$

Backward Euler:

$$x_{n+1} = \frac{x_n}{(1 + h\lambda)}.$$

Trapezoidal:

$$x_{n+1} = \frac{1 - \frac{h\lambda}{2}}{1 + \frac{h\lambda}{2}} x_n.$$

To maintain stability it is important that the values of the equations are such that,

Forward Euler:

$$|1 - h\lambda| \leq 1 \quad \sigma(\theta) \leq 1 - e^{j\theta} \quad 0 \leq \theta \leq 2\pi.$$

Backward Euler:

$$|1 + h\lambda| \leq 1 \quad \sigma(\theta) \geq -1 + e^{j\theta} \quad 0 \leq \theta \leq 2\pi.$$

Trapezoidal:

$$\left| \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right| \leq 1 \quad \sigma(\theta) \leq \frac{-e^{j\theta} + 1}{\frac{1}{2}e^{j\theta} + \frac{1}{2}} \quad 0 \leq \theta \leq 2\pi.$$

The value of $\sigma(\theta)$ corresponds to the polar notation of $h\lambda$, which was achieved through the method of z transforms [5]. The previous equations can be used as a basis for determining the regions of relative stability for each algorithm. Applying similar techniques, it is found that the regions of stability for the other algorithms are as follows,

Simpson's:

$$\sigma(\theta) \leq \frac{-e^{2j\theta} + 1}{\frac{1}{3}(e^{2j\theta} + 4e^{j\theta} + 1)} \quad 0 \leq \theta \leq 2\pi.$$

Parabolic:

$$\sigma(\theta) \leq \frac{e^{2j\theta} - e^{j\theta}}{\frac{1}{12} + \frac{8}{12}e^{j\theta} + \frac{5}{12}e^{2j\theta}} \quad 0 \leq \theta \leq 2\pi.$$

Gear 2nd order:

$$\sigma(\theta) \geq \frac{-e^{2j\theta} + \frac{4}{3}e^{j\theta} - \frac{1}{3}}{\frac{2}{3}e^{2j\theta}} \quad 0 \leq \theta \leq 2\pi.$$

A quick way for determining algorithm stability for a certain problem is to evaluate the above expressions with the proper values of h and λ .

4. Companion circuit techniques

The method illustrated above can now be employed for transient circuit analysis. The equations necessary for the circuit discretization are the terminal relations for inductors and capacitors.

$$v = L \frac{di}{dt} = L i'$$

$$i = C \frac{dv}{dt} = C v'$$

For example, given the inductor in Figure 1., the terminal equation takes the form,

$$v_{(k)} - v_{(m)} = L \frac{di_{(k,m)}}{dt} \quad (1)$$

If Equation (1) is rewritten as an integral, one may use any of the integration techniques mentioned earlier to form the resistive companion circuit. For example, using the trapezoidal method,

$$i_{(k,m)} = \frac{1}{L} \int (v_{(k)} - v_{(m)}) dt$$

now becomes,

$$i_{(k,m) n+1} = i_{(k,m) n} + \frac{h}{2L} [v_{(k,m) n} + v_{(k,m) n+1}] \quad (2)$$

Applying Equation (2), the resistive companion can be solved. Figure 2 shows the example resistive companion for the trapezoidal method.

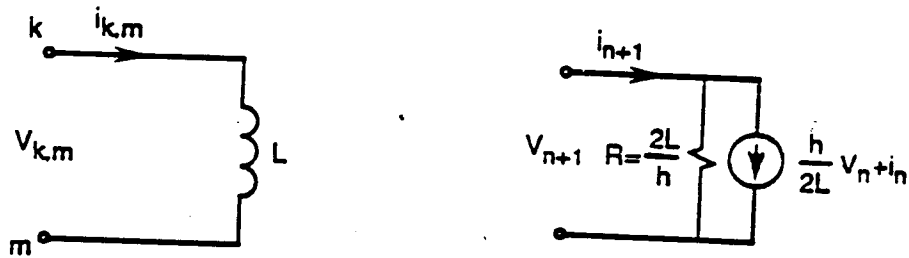


Figure 1 (a) Inductor from node k to node m . (b) Trapezoidal method resistive companion for inductor in (a).

Through similar reasoning, resistive companions are found for each of the methods presented in Section (1). Tables 3 and 4 contain summaries of the relations found between the integration methods and the corresponding companion circuits.

5. Illustration of the technique and error characteristics

By replacing each capacitive and inductive element in a circuit with the corresponding resistive companion, the entire circuit takes the form of purely resistive network. Using Kirchhoff's current law (KCL) for the currents leaving each circuit node, the equations modeling the circuit are written and solved. The most common method for obtaining solutions is to write the equations in matrix format, $I = Y V$ and use LU decomposition to solve for the I vector (I is current, V is node voltage, and Y is the nodal admittance matrix) [6]. For the circuit and input of Figure 2, using the trapezoidal method, one finds the solution shown in Figure 3. For this solution (shown with the solid line), one can calculate an error because the simplicity of the circuit allows calculation of the exact solution. The error magnitude is also shown in Figure 3 as a dotted line. For the same circuit, Figures 4 and 5 show the solution and error characteristics for the parabolic and Gear's 2nd order methods. Additional applications are shown by Thompson in [5].

Table 3 Resistive Companion Circuits for Inductors

Method	Basis of Method	Formula	Resistive Companion
1	Forward Euler	$v_{n+1} = \frac{b}{C} i_n + v_n$	
2	Backward Euler	$i_{n+1} = \frac{C}{h} v_{n+1} - \frac{C}{h} v_n$	
3	Trapezoidal Rule	$i_{n+1} = \frac{3C}{h} v_{n+1} - \frac{2C}{h} v_n - i_n$	
4	Simpson's Rule	$i_{n+1} = \frac{3C}{h} v_{n+1} - \frac{3C}{h} v_{n-1} - 4i_n + i_{n-1}$	
5	Parabolic	$i_{n+1} = \frac{12C}{h} v_{n+1} - \frac{12C}{5h} v_n - \frac{1}{5} i_n + \frac{1}{5} i_{n-1}$	
6	Cubic's 2nd Order	$i_{n+1} = \frac{3C}{2h} v_{n+1} - \frac{2C}{h} v_n + \frac{1}{2h} v_{n-1} + \frac{1}{2h} i_{n-1}$	

Table 4 Resistive Companion Circuits for Capacitors

Method	Basis of Method	Formula	Resistive Companion
1	Forward Euler	$i_{n+1} = \frac{b}{L} v_n + i_n$	
2	Backward Euler	$i_{n+1} = \frac{b}{L} v_{n+1} + i_n$	
3	Trapezoidal Rule	$i_{n+1} = \frac{b}{2L} v_{n+1} + \frac{b}{2L} v_n + i_n$	
4	Simpson's Rule	$i_{n+1} = \frac{b}{3L} v_{n+1} + \frac{4b}{3L} v_n + \frac{b}{3L} v_{n-1} + i_{n-1}$	
5	Parabolic	$i_{n+1} = \frac{5b}{12L} v_{n+1} + \frac{2b}{3L} v_n - \frac{b}{12L} v_{n-1} + i_n$	
6	Cubic's 2nd Order	$i_{n+1} = \frac{2b}{3L} v_{n+1} + \frac{4}{3} i_n - \frac{1}{3} i_{n-1}$	

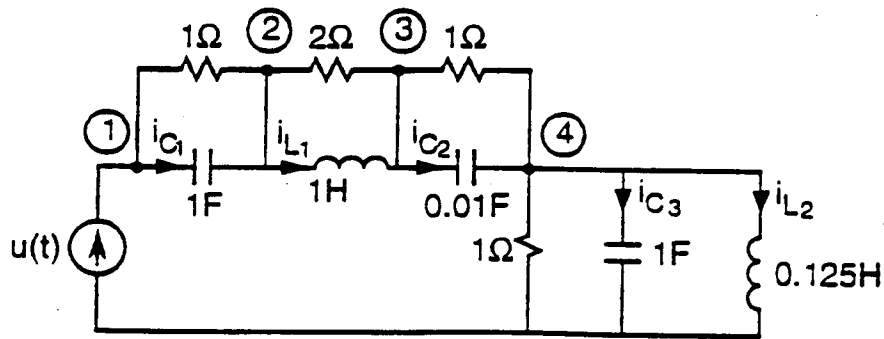


Figure 2 Sample RLC network

6. Conclusions

This paper introduced a number of numerical integration methods which may be used in the resistive companion circuit method of transient circuit analysis. Several of the methods introduced were explored in greater detail, these were the trapezoidal, parabolic, and Gear's 2nd order methods. Of all the methods shown, trapezoidal is the most widely used (it is used in EMTP as well as in SPICE). Reasons for using the trapezoidal method are it's ease of use, its wide range of stability, and it's fairly accurate [1]. However, this paper has shown that other methods (such as the parabolic and Gear's 2nd order methods) do obtain more accurate results than the trapezoidal method (for the same time step). However, the second order methods presented here do have one major drawback -- they all require starting points. The second order methods (since they require the values at time n and $n-1$) cannot begin a solution at time zero because the term at time $n-1$ would occur at negative time.

Of all the methods presented, Simpson's rule showed promise as the most accurate. However, this accuracy cannot be relied on. A quick view at Simpson's region of stability will show that the method is only stable when the value of the time-constant, λ , is imaginary (and this will hardly be the case in electric circuits). The parabolic method shows very good accuracy, but there is a wide region where the parabolic method is unstable. This instability, however, can be countered with effective programming. On the other hand, the Gear's method showed good accuracy and excellent stability. (It should be noted that the Gear's method is also used in SPICE as an alternate to the trapezoidal method).

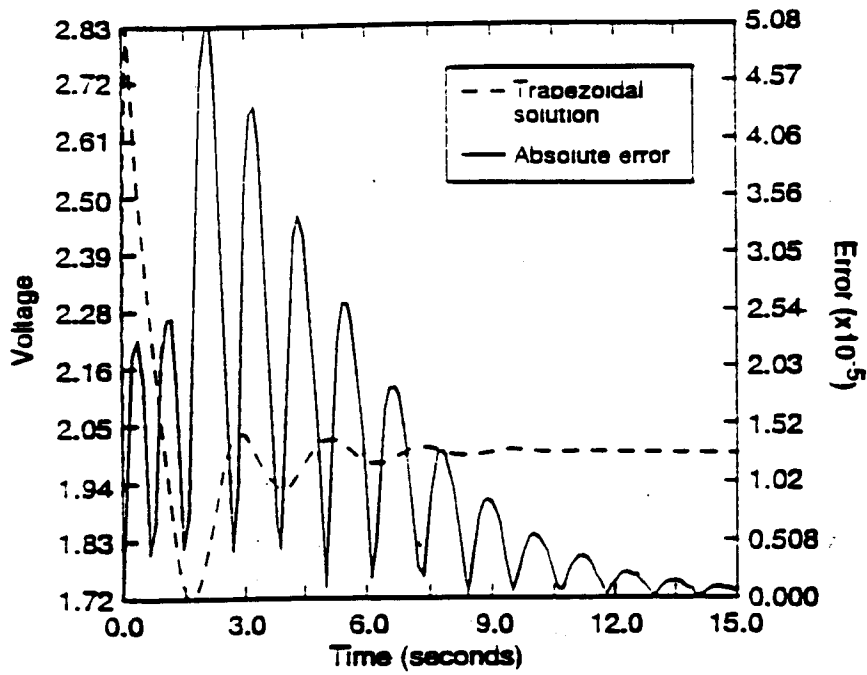


Figure 3 Trapezoidal solution for RLC network ($h = 0.01$ second).

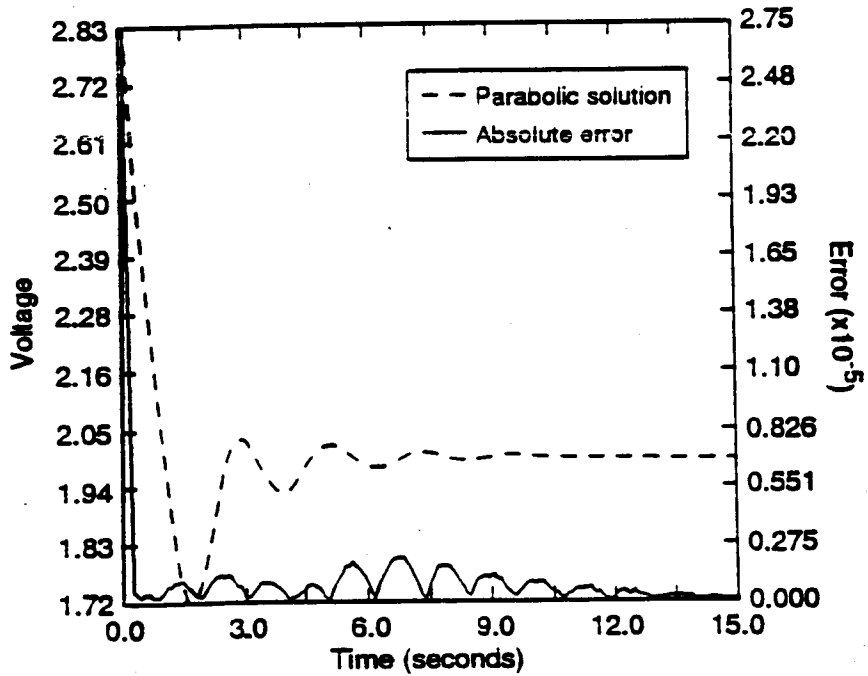


Figure 4 Parabolic solution for RLC network ($h = 0.01$ second).

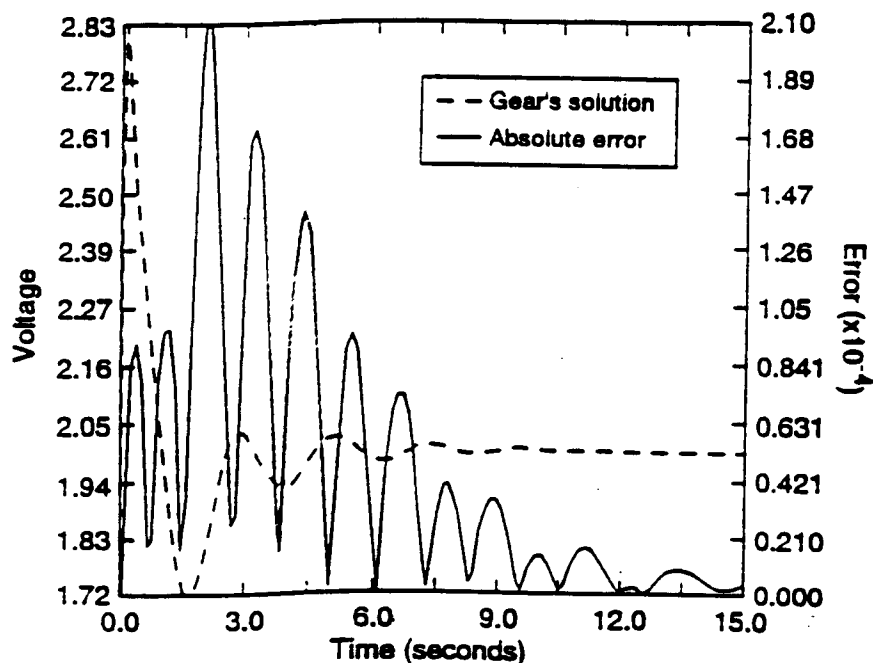


Figure 5 Gear's 2nd order solution to RLC network ($h = 0.01$ second).

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Bibliography

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