

# Fluid Force on a Surface

(Calculate from  $\underline{v}$  and  $p$  fields)

PART II  
LAMINAR  
FLOW  
IN  
A  
TUBE

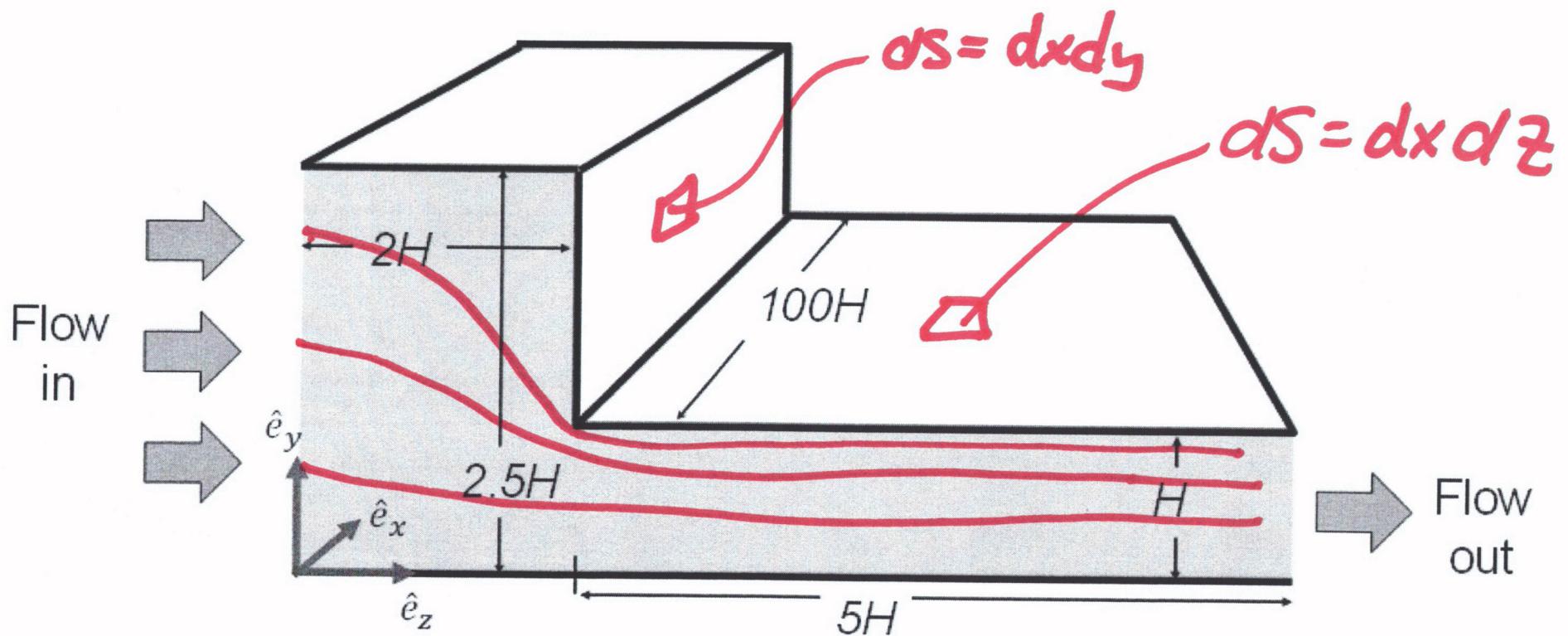
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# What is the force on the top surface?



# Continuum Model of Fluids

Pressure field:  $p = p(x, y, z)$

Velocity field:  $\underline{v}(x, y, z) = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{xyz}$

# Stress $\underline{\underline{\Pi}}$ in the Continuum Model

Fluid stress field:

$$\underline{\underline{\Pi}}(x, y, z) = -p \underline{\underline{I}} + \underline{\underline{\tau}}$$

Newtonian Constitutive Equation:

$$\underline{\underline{\tau}}(x, y, z) = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$$

viscosity

## The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

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### Cartesian Coordinates

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2 \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2 \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

### Cylindrical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tilde{\tau}_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

### Spherical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tilde{\tau}_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \\ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} & 2 \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) \end{pmatrix}_{r\theta\phi}$$


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These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

Where do we get the force/area at each location (due to the fluid flowing)?

$$\underline{\mathcal{F}} = \iint_S (\hat{n} \cdot \underline{\underline{\Pi}})_{at \ surface} dS$$

$$\underline{\underline{\Pi}} = -p \underline{\underline{I}} + \mu(\nabla \underline{\underline{v}} + (\nabla \underline{\underline{v}})^T)$$


**Laminar flow:** What is the fluid force  $\underline{F}$  on the walls of the tube?



$$p = -\frac{\Delta p}{L}z + P_0$$

$$r = R$$

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ \frac{\Delta p R^2}{4\mu L} \left(1 - \frac{r^2}{R^2}\right) \end{pmatrix}_{r\theta z} \approx \underline{\Phi}_0$$

$$\hat{n} = -\hat{e}_r = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}_{r\theta z}$$

$$0 \leq \theta \leq 2\pi \\ 0 \leq z \leq L$$

# Cylindrical

7

$$P = -\frac{\Delta P}{L} z + P_0$$

$$\underline{V} = \begin{pmatrix} 0 \\ 0 \\ \Phi_0 (1 - \frac{r^2}{R^2}) \end{pmatrix}_{r\theta z}$$

$$\begin{aligned} & r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \\ & 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \\ & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ & 2 \frac{\partial v_z}{\partial z} \end{aligned}$$

$$\begin{aligned} \widetilde{\Pi} &= \begin{pmatrix} -p(z) & 0 & 0 \\ 0 & -p(z) & 0 \\ 0 & 0 & -p(z) \end{pmatrix}_{r\theta z} \\ & + \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} \\ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \dots \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial \underline{V}_z}{\partial r} &\neq 0 & \frac{\partial \underline{V}_z}{\partial \theta} &= 0 & \frac{\partial \underline{V}_z}{\partial z} &= 0 \\ \frac{\partial \underline{V}_r}{\partial r} &= \Phi_0 (-\frac{1}{R^2})(2r) \end{aligned}$$

(8)

$$(\hat{n} \cdot \hat{\mathbf{I}}) \Big|_{r=R}$$

$$\begin{pmatrix} -1 & 0 & 0 \end{pmatrix} / -P(z)$$

$$\begin{pmatrix} 0 & -\mu \Phi_0 \frac{z}{R^2} & R \\ 0 & -P(z) & 0 \\ \left( \frac{2\mu \Phi_0}{R^2} \right) r & 0 & -P(z) \end{pmatrix} = \begin{pmatrix} P & 0 & \frac{2\mu \Phi_0}{R} \end{pmatrix}$$

$1 \times 3$

$3 \times 3$

$1 \times 3$

(9)

$$\underline{F} = \iint (\hat{n} \cdot \hat{\underline{H}})_{\text{at surface}} d\underline{s}$$

$$\underline{F} = \iint_0^{2\pi} \left( P(z) \right. \\ \left. \frac{z \mu \Phi_0}{R} \right) R d\theta dz$$

$r\partial z$   
at  $r=R$

(See Part III)