

PART III
LAMINAR
FLOW
(concluded)

Fluid Force on a Surface

(Calculated from \underline{v} and p fields)

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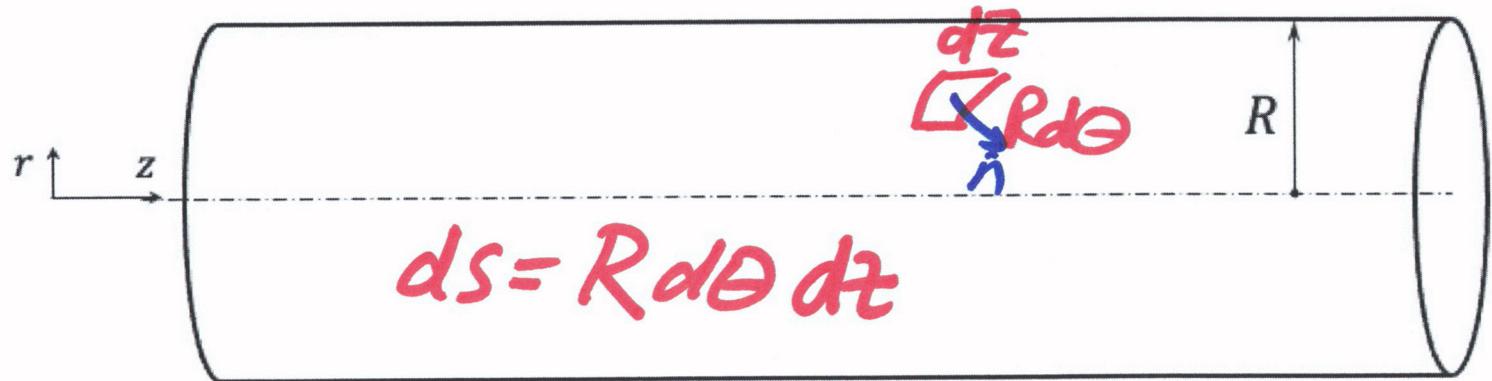
10 December 2020

YouTube: Dr. MorrisonMTU

(pages 1-5, see Part II)

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Laminar flow: What is the fluid force \underline{F} on the walls of the tube?



$$p = -\frac{\Delta p}{L}z + P_0$$

$$r = R$$

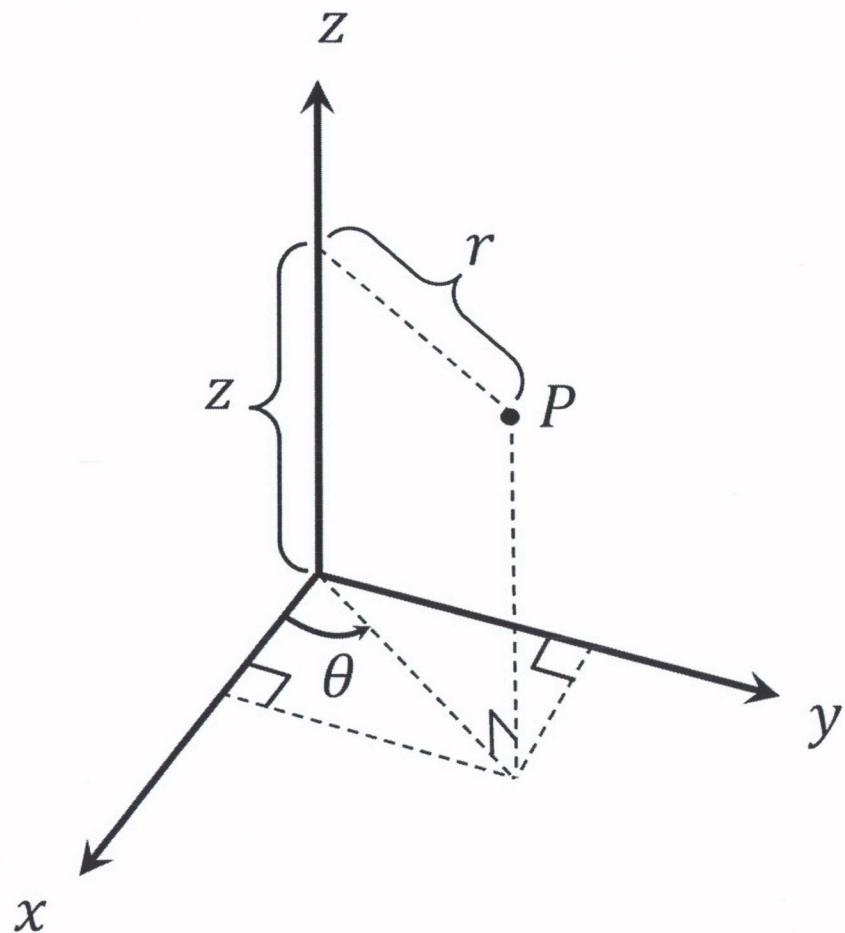
$$\hat{n} = -\hat{e}_r = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}_{r\theta z}$$

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ \frac{\Delta p R^2}{4\mu L} \left(1 - \frac{r^2}{R^2}\right) \end{pmatrix}_{r\theta z}$$

$\underbrace{\quad}_{\equiv} \Phi_0$

$$0 \leq \theta \leq 2\pi$$
$$0 \leq z \leq L$$

Cylindrical Coordinate System: Note that the θ -coordinate swings around the z -axis



Cylindrical

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$$P = -\frac{\Delta P}{L} z + P_0$$

$$\underline{V} = \begin{pmatrix} 0 \\ 0 \\ \Phi_0(1 - \frac{r^2}{R^2}) \end{pmatrix}_{r\theta z}$$

$$\begin{aligned} & r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \\ & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \\ & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ & 2 \frac{\partial v_z}{\partial z} \end{aligned}$$

$$\begin{aligned} \tilde{\Pi} &= \begin{pmatrix} -p(z) & 0 & 0 \\ 0 & -p(z) & 0 \\ 0 & 0 & -p(z) \end{pmatrix}_{r\theta z} \\ & + \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} \\ r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \dots \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial \underline{V}_z}{\partial r} &\neq 0 & \frac{\partial \underline{V}_z}{\partial \theta} &= 0 & \frac{\partial \underline{V}_z}{\partial z} &= 0 \\ \frac{\partial \underline{V}_t}{\partial r} &= \Phi_0 (-\frac{1}{R^2})(2r) \end{aligned}$$

(8)

$$(\hat{n} \cdot \hat{\mathbf{II}}) \Big|_{r=R}$$

$$\begin{pmatrix} -1 & 0 & 0 \end{pmatrix} / -P(1)$$

$$\begin{pmatrix} 0 & -\mu \Phi_0 \frac{2}{R^2} r & R \\ 0 & -P(2) & 0 \\ \left(\frac{2\mu \Phi_0}{R^2}\right)r & 0 & -P(2) \end{pmatrix}$$

$$3 \times 3$$

$$= \begin{pmatrix} P & 0 & \frac{2\mu \Phi_0}{R^2} \end{pmatrix}$$

$$1 \times 3$$

$$1 \times 3$$

(9)

$$\underline{F} = \iint (\hat{n} \cdot \underline{\underline{\Pi}})_{\text{at surface}} d\underline{s}$$

$$\underline{F} = \iint_0^{2\pi} \left(P(z) \right. \\ \left. \frac{z\mu\Phi_0}{R} \right) R d\theta dz$$

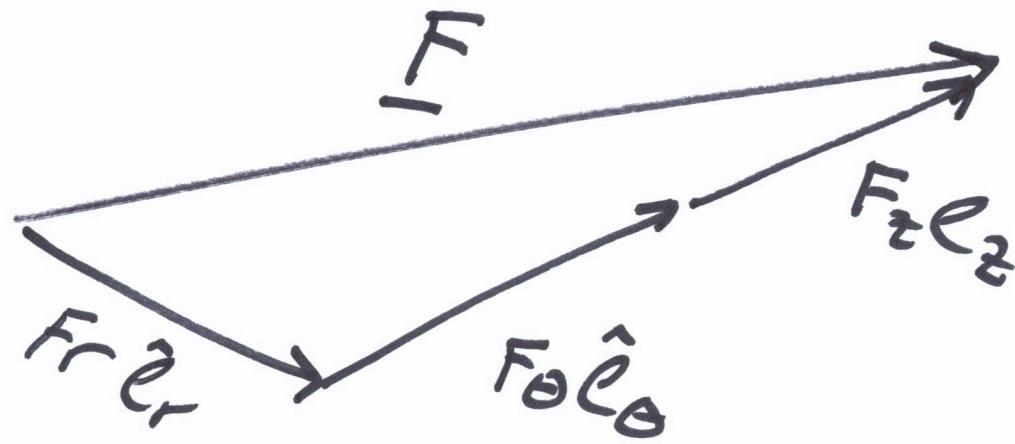
$r\theta z$
at $r=R$

(See Part III)

Part III begins...

(10)

$$\underline{F} = \begin{pmatrix} F_r \\ F_\theta \\ F_z \end{pmatrix}_{r\theta z} = F_r \hat{e}_r + F_\theta \hat{e}_\theta + F_z \hat{e}_z$$



z-component of \vec{F}

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$$F_z \hat{\epsilon}_z = \int_0^L \int_0^{2\pi} \hat{\epsilon}_z \frac{2\Delta}{R} \Phi_0 R d\theta dz$$

$\frac{\Delta P R^2}{4\mu L}$

$$= \hat{\epsilon}_z \int_0^L \int_0^{2\pi} \left(\frac{8 \Delta P R^2}{4L} \right) d\theta dz$$

$$\int_0^{2\pi} d\theta = \theta \Big|_0^{2\pi} = 2\pi$$

$$\int_0^L dz = z \Big|_0^L = L$$

$$= \hat{\epsilon}_z \cancel{2\pi L} \frac{\Delta P R^2}{2\Delta} = \boxed{\underline{\pi R^2 \Delta P \hat{\epsilon}_z = F_z \hat{\epsilon}_z}}$$

θ -component of \underline{F}

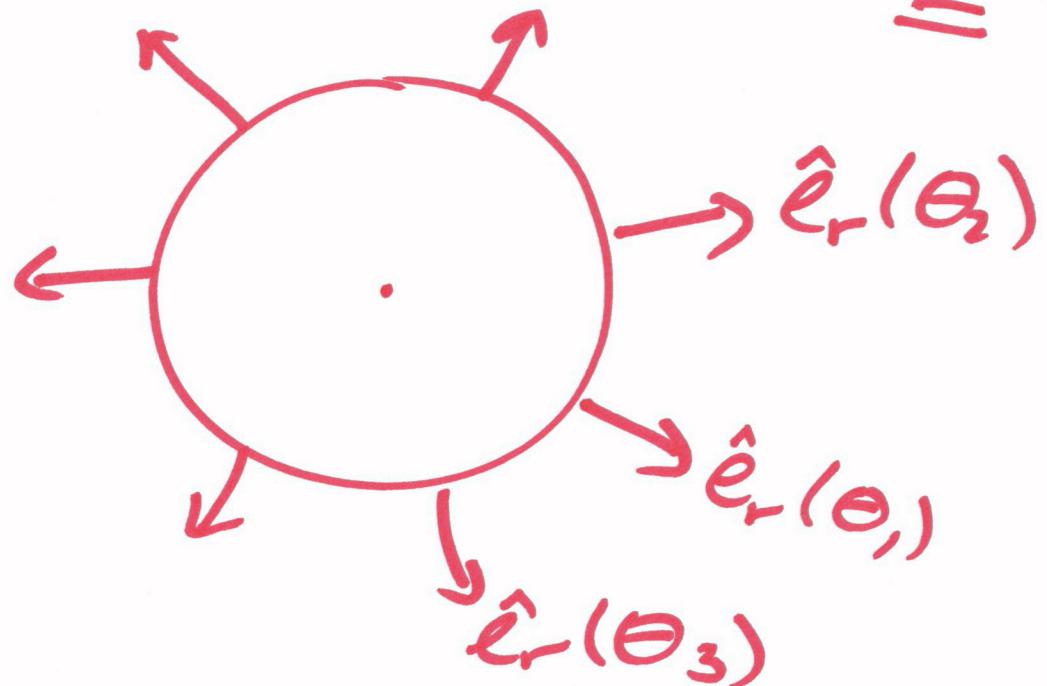
zero!

$$F_\theta \hat{e}_\theta = \iint \hat{e}_\theta \underline{U} R d\theta dz = 0$$

r -component of \underline{F}

$$F_r \hat{e}_r = \int_0^L \int_0^{2\pi} \left(-\left(\frac{P_0 - P_L}{L} z + P_0 \right) \hat{e}_r R d\theta dz \right)$$

Variable!
↓



notk: $\hat{e}_r = \cos\theta \hat{e}_x + \sin\theta \hat{e}_y$

(FROM
GEOMETRY OF
COORD SYSTEM)

\hat{e}_x, \hat{e}_y are constants

$$F_r \hat{e}_r = R \int_0^L \int_0^\pi \left[\frac{P_L - P_0}{L} z + P_0 \right] (\underbrace{\cos\theta \hat{e}_x + \sin\theta \hat{e}_y}_{\hat{e}_r})$$

z-integral:

$$\int_0^L \left(\left(\frac{P_L - P_0}{L} \right) z + P_0 \right) dz \quad d\theta dz$$

$$= \left(\frac{P_L - P_0}{L} \right) \frac{z^2}{2} + P_0 z \Big|_0^L$$

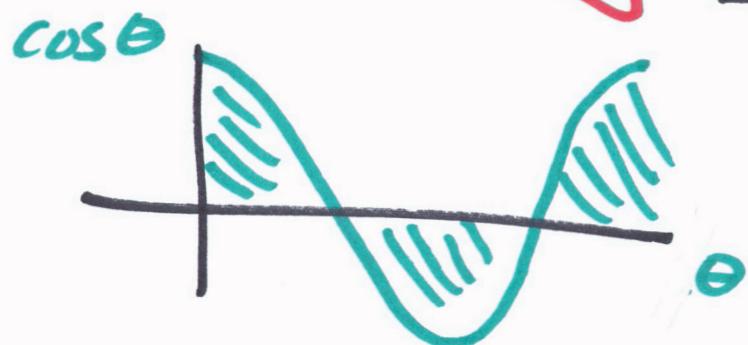
$$= L \left[\frac{P_L}{2} - \frac{P_0}{2} + P_0 \right]$$

$$= L \left(\frac{P_0 + P_L}{2} \right)$$

θ -integral:

$$F_r \hat{e}_r = R \int_0^{2\pi} \left(\frac{c}{2} (P_0 + P_L) \right) [\cos \theta \hat{e}_x + \sin \theta \hat{e}_y] d\theta$$

$$= \frac{RL}{2} (P_0 + P_L) \left[\hat{e}_x \int_0^{2\pi} \cos \theta d\theta + \hat{e}_y \int_0^{2\pi} \sin \theta d\theta \right]$$



$$\frac{F_r \hat{e}_\theta}{r} = 0.$$

$$\underline{F} = \begin{pmatrix} F_r \\ F_\theta \\ F_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} 0 \\ 0 \\ \underbrace{\Delta P \pi R^2}_{r\theta z} \end{pmatrix}$$

This is a
general result:

$$F_z = F_{dry} = \Delta P A_{cross \text{ section}}$$