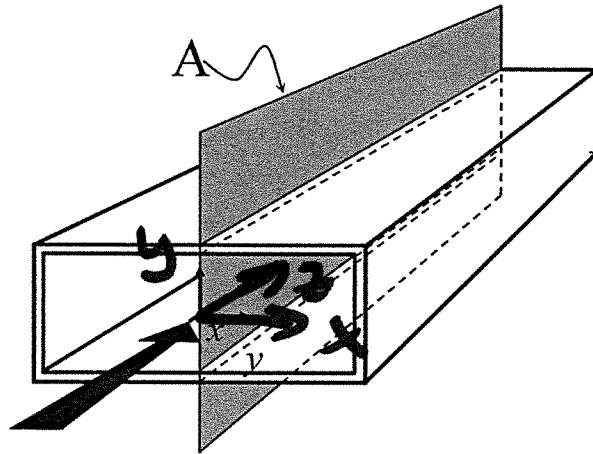


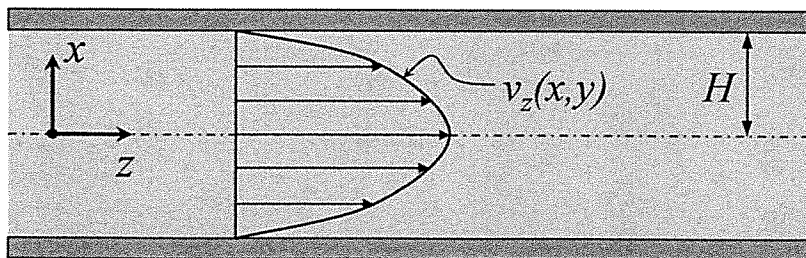


### Example III: Pressure-driven flow of a Newtonian fluid in a rectangular duct: Poiseuille flow



What is the steady state velocity profile for laminar flow of an incompressible Newtonian fluid flowing down a long duct of rectangular cross section? The duct height is  $2H$  and the width is  $2W$ . The pressure at an upstream position is  $p_0$  and a distance  $L$  downstream the pressure is  $p_L$ . What is the force on the walls?

cross-section A:



See hand notes

$$v_z = v_z(x, y)$$

(Example 7.11, p549)

2

$\rho = \rho(r, \theta, z, t)$

# The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

$$\vec{v} = (0, 0, v_z)$$

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

$$\frac{\partial v_z}{\partial z} = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_z)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left( \frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z} \right) + \rho g_x \\ \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left( \frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\tau}_{yy}}{\partial y} + \frac{\partial \bar{\tau}_{zy}}{\partial z} \right) + \rho g_y \\ \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left( \frac{\partial \bar{\tau}_{xz}}{\partial x} + \frac{\partial \bar{\tau}_{yz}}{\partial y} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \left( \frac{1}{r} \frac{\partial(r \bar{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta r}}{\partial \theta} - \frac{\bar{\tau}_{\theta\theta}}{r} + \frac{\partial \bar{\tau}_{zr}}{\partial z} \right) + \rho g_r \\ \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left( \frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{\tau}_{z\theta}}{\partial z} + \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} \right) + \rho g_\theta \\ \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left( \frac{1}{r} \frac{\partial(r \bar{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\begin{aligned} &\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2}{r} + \frac{v_\phi^2}{r} \right) \\ &= -\frac{\partial P}{\partial r} + \left( \frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta r} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi r}}{\partial \phi} - \frac{\bar{\tau}_{\theta\theta}}{r} + \frac{\bar{\tau}_{\phi\phi}}{r} \right) + \rho g_r \\ &\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} + \frac{v_\phi^2 \cot \theta}{r} \right) \\ &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left( \frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi\theta}}{\partial \phi} + \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} - \frac{\bar{\tau}_{\phi\phi} \cot \theta}{r} \right) + \rho g_\theta \\ &\rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right) \\ &= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \left( \frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta\phi} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi\phi}}{\partial \phi} + \frac{\bar{\tau}_{\phi r} - \bar{\tau}_{r\phi}}{r} + \frac{\bar{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\phi \end{aligned}$$

(rectangular duct)

$v_x = 0$   
 $v_y = 0$   
 $\frac{dv_z}{dt} = 0$  (continuity)

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho \left( \frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial x} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

$$\rho \left( \frac{\partial v_y}{\partial t} + v_r \frac{\partial v_y}{\partial x} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rv_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2}{r} - \frac{v_\phi^2}{r} \right) = -\frac{\partial P}{\partial r} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_r$$

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} + \frac{v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta$$

$$\rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right) + \rho g_\phi$$

Note: the  $r$ -component of the Navier-Stokes equation in spherical coordinates may be simplified by adding  $0 = \frac{2}{r} \nabla \cdot \underline{v}$  to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2<sup>nd</sup> edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

# Rectangular duct

3

x-component NS

$$0 = -\frac{\partial P}{\partial x} - \rho g$$

y-component NS

$$0 = -\frac{\partial P}{\partial y}$$

z-component of NS

$$0 = -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

$$\underbrace{\frac{\partial P}{\partial z}}_{\text{function of } z} = \underbrace{\mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)}_{\text{function of } x \text{ and } y} = \underbrace{\quad}_{\text{equal to const.}}$$

5

LHS: solves simply

RHS: HARD

(see book)

Force on walls?

$F_z$

What is  $F_z$  ?

$$F_z = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{x \times z}$$

$$= (\hat{e}_z \cdot \underline{F})$$

What is  $\underline{F}$  ?

7

$$E = \iint_S (\hat{n} \cdot \vec{\pi})_{\text{at surface}} ds$$

$$\vec{\pi} = ?$$

$$\hat{n} = ?$$

$$\underline{\underline{\tilde{u}}} = -P \underline{\underline{I}} + \underline{\underline{\tilde{u}}}$$

$$-P \underline{\underline{I}} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}_{\text{real}}$$

$\underline{\underline{\tilde{u}}}$  ? from  $\underline{\underline{v}}$   
and handout  $\downarrow$



# The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Prof Faith A. Morrison, Michigan Technological University  
**BEFANVSURK DUCT** (15/15)

$U = 10$   
 $z$

## Cartesian Coordinates

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \mu \begin{pmatrix} \frac{2}{\partial x} \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2 \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}$$

## Cylindrical Coordinates

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix} = \mu \begin{pmatrix} \frac{2}{\partial r} \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}$$

## Spherical Coordinates

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \tau_{\phi\phi} \end{pmatrix} = \mu \begin{pmatrix} \frac{2}{\partial r} \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \\ \frac{r}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} & 2 \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) \end{pmatrix}$$

These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

9.5

$$\tilde{\Pi} = -P\tilde{I} + \tilde{C}$$

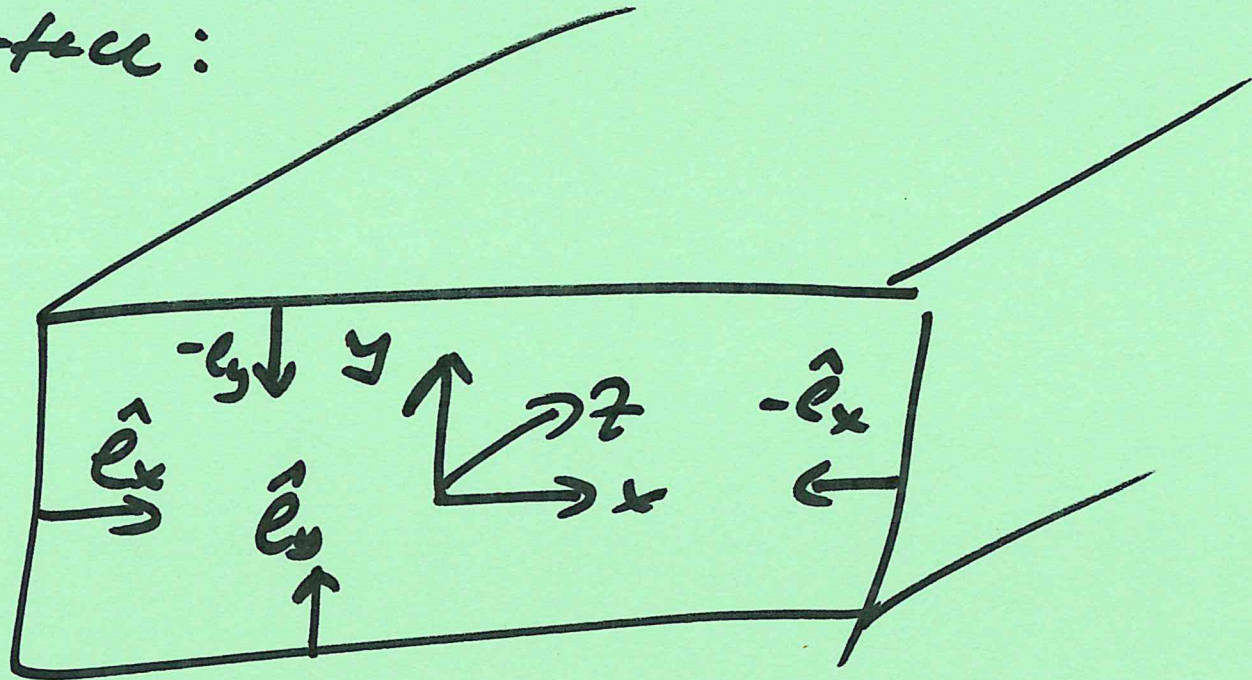
$$= \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}_{x,y,z} + \begin{pmatrix} 0 & 0 & \mu \frac{\partial v_z}{\partial x} \\ \mu \frac{\partial v_z}{\partial x} & 0 & \mu \frac{\partial v_z}{\partial y} \\ \mu \frac{\partial v_z}{\partial y} & \mu \frac{\partial v_z}{\partial y} & 0 \end{pmatrix}$$

$$P = P(z)$$

$$= \begin{pmatrix} -P & 0 & \mu \frac{\partial v_z}{\partial x} \\ 0 & -P & \mu \frac{\partial v_z}{\partial y} \\ \mu \frac{\partial v_z}{\partial x} & \mu \frac{\partial v_z}{\partial y} & -P \end{pmatrix}_{x,y,z}$$

$\hat{n}$  depends  
on the surface:

(10)



→  $(\hat{n} \cdot \underline{\underline{\Gamma}})$

calc for each  
surface  
+ add up //