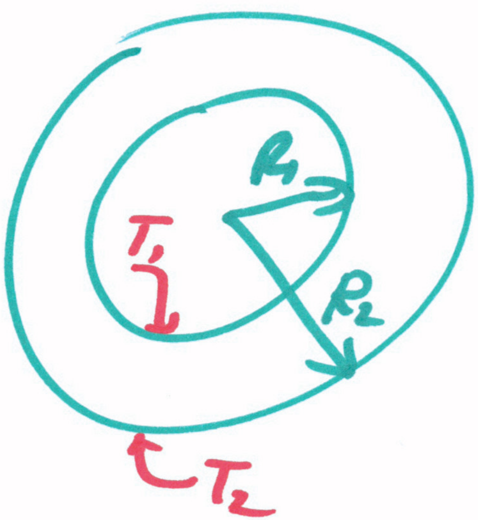
Microscopic Energy Balance in a Tube (1D Radial Heat Conduction)



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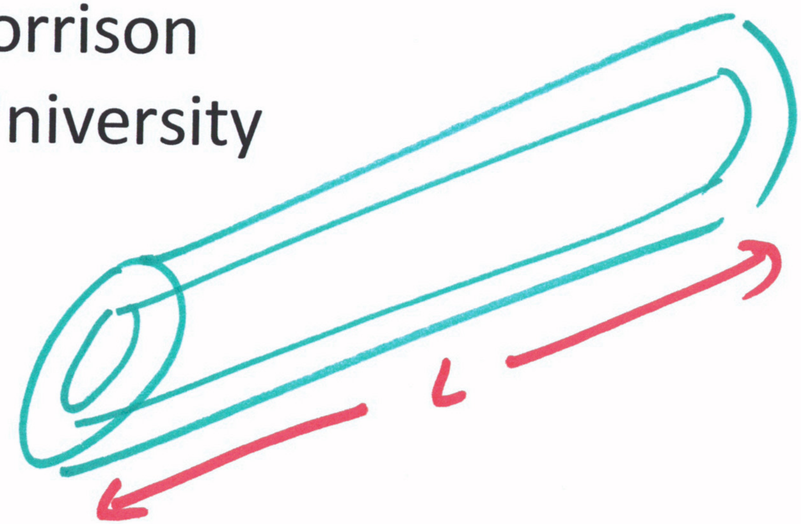
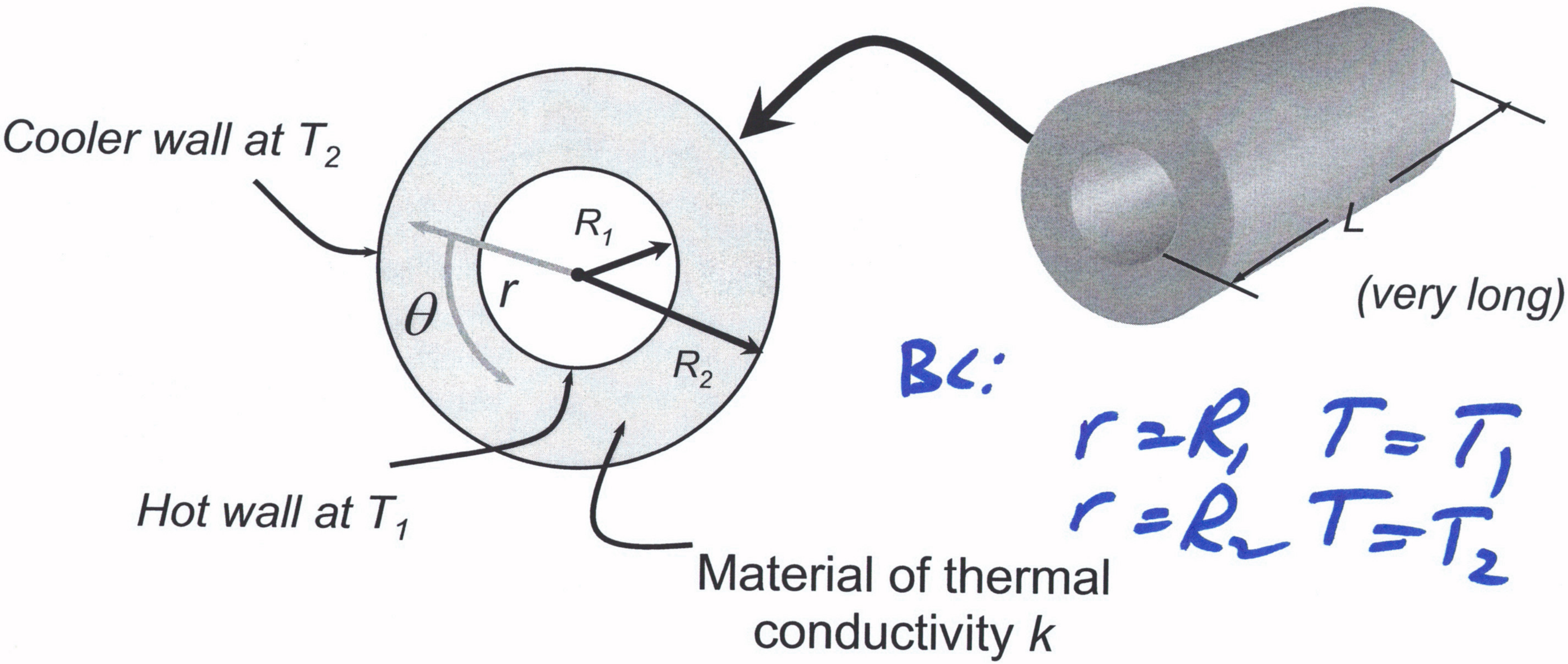


16 December 2020

YouTube: https://youtu.be/10dYOQXqI4E

Example: *What is the steady state temperature profile in a long cylindrical shell (pipe) if the* ***inner wall*** *is held at T_1 and the* ***outer wall*** *is held at T_2 ($T_1 > T_2$)? What is the heat flux in the shell?*

$R_1 \le r \le R_2$



BC:

$r = R_1 \quad T = T_1$

$r = R_2 \quad T = T_2$

pages.mtu.edu/~fmorriso/cm310/energy.pdf

The **Equation of Energy** in Cartesian, cylindrical, and spherical coordinates for Newtonian fluids of constant density, with source term S_e. Source could be electrical energy due to current flow, chemical energy, etc. Two cases are presented: the general case where thermal conductivity may be a function of temperature (vector flux $\underline{\tilde{q}} = \underline{q}/area$ appears in the equations); and the more usual case, where thermal conductivity is constant (on the reverse).

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Microscopic energy balance, in terms of flux; Gibbs notation

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = -\nabla\cdot\underline{\tilde{q}} + S_e$$

Microscopic energy balance, in terms of flux; Cartesian coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = -\left(\frac{\partial\tilde{q}_x}{\partial x} + \frac{\partial\tilde{q}_y}{\partial y} + \frac{\partial\tilde{q}_z}{\partial z}\right) + S_e$$

Microscopic energy balance, in terms of flux; cylindrical coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial\theta} + v_z\frac{\partial T}{\partial z}\right) = -\left(\frac{1}{r}\frac{\partial(r\tilde{q}_r)}{\partial r} + \frac{1}{r}\frac{\partial\tilde{q}_\theta}{\partial\theta} + \frac{\partial\tilde{q}_z}{\partial z}\right) + S_e$$

Microscopic energy balance, in terms of flux; spherical coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial\phi}\right) = -\left(\frac{1}{r^2}\frac{\partial(r^2\tilde{q}_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tilde{q}_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tilde{q}_\phi}{\partial\phi}\right) + S_e$$

Fourier's law of heat conduction, Gibbs notation: $\underline{\tilde{q}} = \underline{q}/A = -k\nabla T$

Fourier's law of heat conduction, Cartesian coordinates:
(constant thermal conductivity k)

$$\begin{pmatrix}\tilde{q}_x\\ \tilde{q}_y\\ \tilde{q}_z\end{pmatrix}_{xyz} = \begin{pmatrix}q_x/A\\ q_y/A\\ q_z/A\end{pmatrix}_{xyz} = \begin{pmatrix}-k\frac{\partial T}{\partial x}\\ -k\frac{\partial T}{\partial y}\\ -k\frac{\partial T}{\partial z}\end{pmatrix}_{xyz}$$

Fourier's law of heat conduction, cylindrical coordinates:
(constant thermal conductivity k)

$$\begin{pmatrix}\tilde{q}_r\\ \tilde{q}_\theta\\ \tilde{q}_z\end{pmatrix}_{r\theta z} = \begin{pmatrix}q_r/A\\ q_\theta/A\\ q_z/A\end{pmatrix}_{r\theta z} = \begin{pmatrix}-k\frac{\partial T}{\partial r}\\ -\frac{k}{r}\frac{\partial T}{\partial\theta}\\ -k\frac{\partial T}{\partial z}\end{pmatrix}_{r\theta z}$$

Fourier's law of heat conduction, spherical coordinates:
(constant thermal conductivity k)

$$\begin{pmatrix}\tilde{q}_r\\ \tilde{q}_\theta\\ \tilde{q}_\phi\end{pmatrix}_{r\theta\phi} = \begin{pmatrix}q_r/A\\ q_\theta/A\\ q_\phi/A\end{pmatrix}_{r\theta\phi} = \begin{pmatrix}-k\frac{\partial T}{\partial r}\\ -\frac{k}{r}\frac{\partial T}{\partial\theta}\\ -\frac{k}{r\sin\theta}\frac{\partial T}{\partial\phi}\end{pmatrix}_{r\theta\phi}$$

Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term. Source could be

1. viscous dissipation due to vigorous flow,
2. electrical energy due to electric current,
3. chemical energy due to reaction, etc., with units of energy/(volume time)).

Cylindrical coordinates $(r\theta z)$:

no elec current
no rxn

$$\cancel{\frac{\partial T}{\partial t}} + \cancel{v_r \frac{\partial T}{\partial r}} + \cancel{\frac{v_\theta}{r}} \frac{\partial T}{\partial r} + \cancel{v_z \frac{\partial T}{\partial z}} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right) + \cancel{\frac{S_e}{\rho \hat{C}_p}}$$

steady

v=0

θ symmetry

long tube

③

$$0 = \frac{\cancel{k}}{\cancel{\rho c_p}}\left(\frac{1}{\cancel{r}}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right)$$

$$\frac{d}{dr}\underbrace{\left(r\frac{dT}{dr}\right)}_{\equiv \Phi} = 0$$

$$\frac{d\Phi}{dr} = 0$$

integrate:

$$r\frac{dT}{dr} = \Phi = c_1$$

$$\frac{dT}{dr} = \frac{c_1}{r}$$

integrate:

$$\boxed{T = c_1 \ln r + c_2}$$

logarithmic

Flux:

$$\boxed{\frac{q_r}{A} = -k\frac{dT}{dr} = -k\left(\frac{c_1}{r}\right)}$$

goes like $\frac{1}{r}$