Nondimensionalize Semi-infinite slab problem

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Choice:

For the unsteady case we'll choose a characteristic time based on the thermal diffusivity, α.

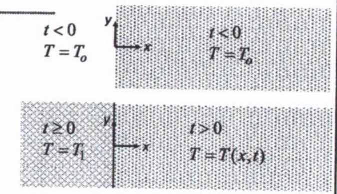
$$\frac{t}{t_{char}} = t^* \equiv \underbrace{\frac{\alpha t}{D_{char}^2}}$$

This dimensionless time is called Fourier number Fo.

$$t_{char} = \frac{D^2}{\alpha} = \frac{D^2 \rho \hat{C}_p}{k}$$

(thermal diffusion)

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\boxed{\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha \left(\frac{\partial^2 T}{\partial x^2}\right)}$$

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

Initial condition: $t = 0 \quad T = T_0 \quad \forall\ x$

Boundary conditions:

$$x = 0 \quad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \quad t > 0$$

$$x = \infty \quad T = T_0 \quad \forall\ t$$

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

(Appears in the energy balance)

$$\frac{D^2}{\alpha} = \text{thermal diffusion time} = \frac{D^2 \rho \hat{C}_p}{k}$$

1D Heat Transfer: Unsteady State

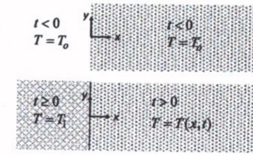
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Let's do it.

Dimensional Analysis, Unsteady State Convection

Non-dimensionalize (eqns, BCs)

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right) \qquad q_x = -k\frac{\partial T}{\partial x} = hA(T_1 - T)$$



non-dimensional variables:

position:	temperature:	time:
$x^* \equiv \dfrac{x}{D}$	$Y \equiv \dfrac{(T_1 - T)}{(T_1 - T_0)}$	$t^* \equiv \dfrac{\alpha t}{D^2}$

This dimensionless time is called Fourier number Fo.

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial t^*}\frac{\partial t^*}{\partial t} = \boxed{\frac{\partial T}{\partial t^*}\frac{\alpha}{D^2}}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial x^*}\frac{\partial x^*}{\partial x} = \frac{\partial T}{\partial x^*}\frac{1}{D}$$

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$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \underbrace{\frac{\partial T/\partial x}{\partial x^*}}\;\underbrace{\frac{\partial x^*}{\partial x}}_{\frac{1}{D}}$$

$$\underbrace{\frac{\partial}{\partial x^*}\frac{\partial T}{\partial x}} = \underbrace{\frac{\partial}{\partial x^*}\frac{\partial T}{\partial x^*}}_{\frac{\partial^2 T}{\partial x^{*2}}}\frac{1}{D}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial x^{*2}}\frac{1}{D^2}}$$

$$T^* = \left(\frac{T_1 - T}{T_1 - T_0}\right) = \frac{T_1}{T_1 - T_0} - \left(\frac{1}{T_1 - T_0}\right) T$$

$$\frac{\partial T^*}{\partial t^*} = -\left(\frac{1}{T_1 - T_0}\right) \underbrace{\frac{\partial T}{\partial t^*}}_{\text{we need this above;}}$$

solve for this + substitute

From above: $$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial t^*} \frac{\alpha}{D^2}$$

$$\boxed{\frac{\partial T}{\partial t} = -(T_1 - T_0) \frac{\partial T^*}{\partial t^*} \frac{\alpha}{D^2}}$$

LHS of differential eqn

RHS:

$$\frac{\partial T^*}{\partial x^*} = -\frac{1}{(T_1 - T_0)} \frac{\partial T}{\partial x^*}$$

$$\frac{\partial}{\partial x^*}\left(\frac{\partial T^*}{\partial x^*}\right) = -\frac{1}{(T_1 - T_0)} \underbrace{\frac{\partial^2 T}{\partial x^*}}$$

we need this above (p3); solve for this + substitute

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial x^{*2}} \frac{1}{D^2}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T^*}{\partial x^{*2}} (-1)(T_1 - T_0) \frac{1}{D^2}}$$

on RHS of differential eqn

Substitute:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\cancel{1}\,(\cancel{T_1 - T_0})\,\frac{\partial T^*}{\partial t^*}\,\cancel{\frac{\alpha}{D^2}} = \cancel{\frac{\alpha}{D^2}}\,(\cancel{1})\,(\cancel{T_1 - T_0})\,\frac{\partial^2 T^*}{\partial x^{*2}}$$

$$\boxed{\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}}$$

(no dimensionless groups appear)

Now, boundary conditions $\longrightarrow$

Initial Condition:

$t = 0 \quad T = T_0 \quad \forall x$

$$\boxed{t^* = 0 \quad T^* = 1 \quad \forall x^*}$$

BC1:

$x = 0 \quad -k \underbrace{\frac{dT}{dx}} = h(\overbrace{T_1 - T}^{T^*(T_1 - T_0)}) \quad \forall t > 0$

$= \frac{\partial T}{\partial x^*} \frac{1}{D}$

$= \left(\frac{\partial T^*}{\partial x^*}\right)(-1)(T_1 - T_0)\frac{1}{D}$

$x^* = 0$

$t^* > 0$

$\cancel{-}k\left(\frac{\partial T^*}{\partial x^*}\right) \frac{\cancel{-}1}{D}(\cancel{T_1 - T_0})$

$= h T^* (\cancel{T_1 - T_0})$

$$\boxed{\frac{\partial T^*}{\partial x^*} = \frac{hD}{k} T^*}$$

BC 2:

$x = \infty \quad T = T_0 \quad \forall t \longrightarrow \boxed{x^* = \infty \quad T^* = 1 \quad \forall t}$

Final Version:

$$\boxed{\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}}$$

$t^* = 0 \quad T^* = 1 \quad \forall x^*$

$x^* = 0 \quad \frac{\partial T^*}{\partial x^*} = \frac{hD}{k} T^* \quad \forall t^* > 0$

Biot #

$x^* = \infty \quad T^* = 1 \quad \forall t$

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