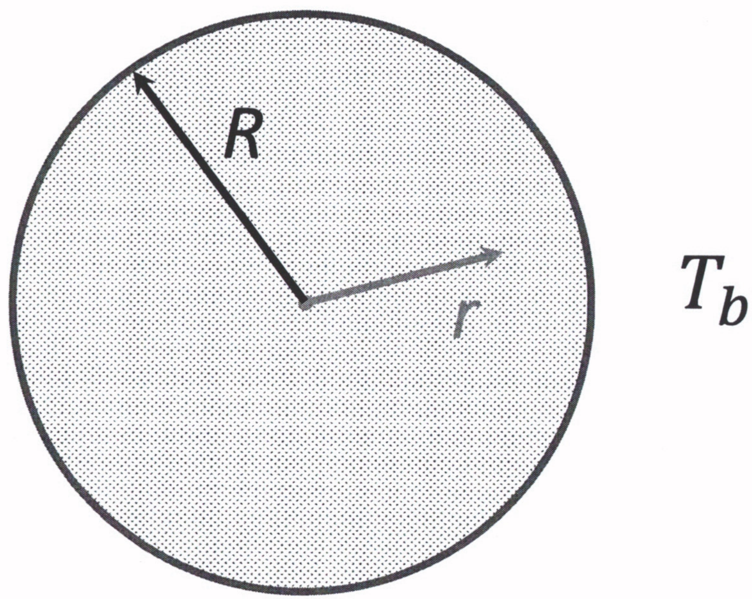
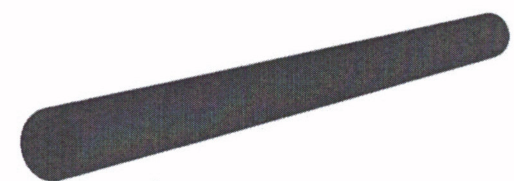
Example 3: Heat Conduction with Generation

What is the steady state temperature profile in a wire if heat is generated uniformly throughout the wire at a rate of S_e W/m³ and the bulk fluid surrounding the wire is at T_b? What is the heat flux?





S_e = energy production per unit volume

Boundaries: $r = 0$
$r = R$

②

The **Equation of Energy** for systems with **constant k**

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t}+\underline{v}\cdot\nabla T\right)=k\nabla^2 T+S_e$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+S_e$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho\hat{C}_p\left(\cancel{\frac{\partial T}{\partial t}}+\cancel{v_r\frac{\partial T}{\partial r}}+\cancel{\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}}+\cancel{v_z\frac{\partial T}{\partial z}}\right)=k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}}+\cancel{\frac{\partial^2 T}{\partial z^2}}\right)+S_e$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$
$$=k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right)+S_e$$

long wire

θ symmetry

$\underline{v}=0$

steady

$$0=k\,\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+S_e$$

Reference: F. A. Morrison, "Web Appendix to *An Introduction to Fluid Mechanics*," Cambridge University Press, New York, 2013 (*pages.mtu.edu/~fmorriso/cm310/IFMWebAppendixDMicroEBalanceMorrison.pdf*). This worksheet is on the web at *pages.mtu.edu/~fmorriso/cm310/energy.pdf*.

$$-\frac{S_e}{k} r = \frac{d}{dr}\underbrace{\left(r\frac{dT}{dr}\right)}_{\equiv \Phi} = \frac{d\Phi}{dr}$$

$$\frac{d\Phi}{dr} = \left(\frac{-S_e}{k}\right) r$$

$$r\frac{dT}{dr} = \Phi = \left(\frac{-S_e}{k}\right)\frac{r^2}{2} + C_1$$

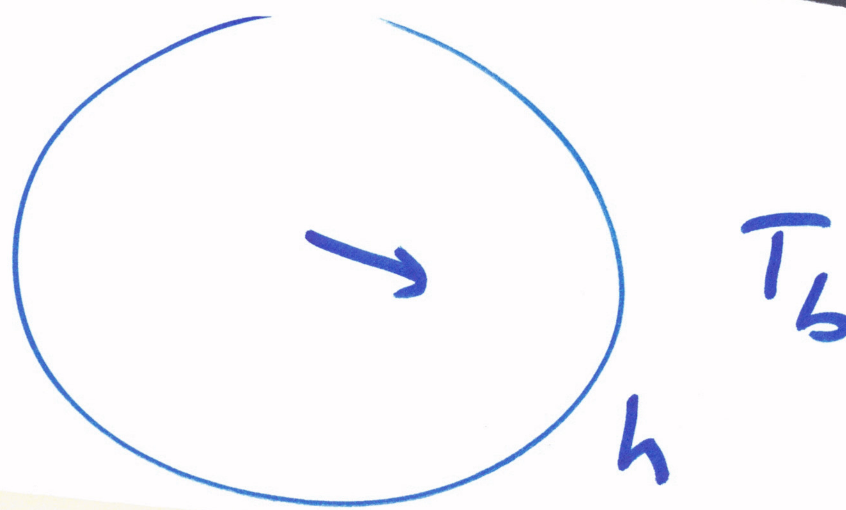
$$\frac{dT}{dr} = \frac{r}{2}\left(\frac{-S_e}{k}\right) + \frac{C_1}{r}$$

$$\boxed{T = \left(\frac{-S_e}{2k}\right)\frac{r^2}{2} + C_1 \ln r + C_2}$$

Boundary Conditions

$r = 0 \qquad T = \text{finite} \qquad \boxed{c_1 = 0}$

$r = R$



Newton's Law of cooling

$$-k \frac{dT}{dr} = \boxed{\frac{q_r}{A} = h\left(T(R) - T_b\right)}$$

. . . solve

$$T = \left(\frac{-S_e}{4k}\right) r^2 + C_2$$

$$\frac{dT}{dr} = \left(\frac{-S_e}{\cancel{4}_{2} k}\right)(\cancel{2} r)$$

$$-k \left.\frac{dT}{dr}\right|_{r=R} = h\left(T(R) - T_b\right)$$

$$\frac{\cancel{-}k}{h}\left(\cancel{-}\frac{S_e}{2k} R\right) = \frac{\cancel{h}}{\cancel{h}}\left[-\frac{S_e}{4k} R^2 + C_2 - T_b\right]$$

$$\boxed{C_2 = \frac{\cancel{k}}{h}\frac{S_e}{2\cancel{k}} R + \frac{S_e}{4k} R^2 + T_b}$$

$$T = \left(-\frac{S_e}{4k}\right) r^2 + \frac{S_e R}{2h} + \frac{S_e R^2}{4k}$$

$$\boxed{T - T_b = \frac{S_e}{4k}\left[R^2 - r^2\right] + \frac{S_e R}{2h}}$$