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CM3120

We have solved the "freezing pipe" problem. Now we non-dimensionalize both the equation + the BC, IC.

We hope to identify dimensionless parameters that will help us map out regimes of unsteady heat conduction behavior.

non dimensionalize the eqn:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$Y = T^* = \left(\frac{T_1 - T}{T_1 - T_0}\right)$$

$$Y(T_1 - T_0) = T_1 - T$$

solve for T:

$$T = T_1 - Y(T_1 - T_0)$$

take t deriv:

$$\frac{\partial T}{\partial t} = -\frac{\partial Y}{\partial t}(T_1 - T_0)$$

dimensionless time:

$$t^* = \frac{\alpha t}{D^2}$$

take t deriv:

$$\frac{\partial t^*}{\partial t} = \frac{\alpha}{D^2}$$

we want to substitute here:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

chain rule:

$$\frac{\partial T}{\partial t^*} \underbrace{\frac{\partial t^*}{\partial t}}_{\downarrow} = \frac{\partial T}{\partial t}$$

$$\underbrace{\frac{\partial T}{\partial t^*}} \left(\frac{\alpha}{D^2}\right) = \frac{\partial T}{\partial t} \Rightarrow \boxed{\frac{\partial T}{\partial t^*} = \frac{D^2}{\alpha} \frac{\partial T}{\partial t}}$$

what I want is $\frac{\partial Y}{\partial t^*}$

will use next page

$$Y(T_1 - T_0) = T_1 - T$$

take deriv. wrt t^*

result of the derivative:

$$(T_1 - T_0)\frac{\partial Y}{\partial t^*} = (-1)\underbrace{\frac{\partial T}{\partial t^*}}_{\frac{D^2}{\alpha}\frac{\partial T}{\partial t}}$$

(see previous page)

solve for $\frac{\partial Y}{\partial t^*}$:

$$\frac{\partial Y}{\partial t^*} = \left(\frac{-1}{T_1 - T_0}\right)\frac{D^2}{\alpha}\frac{\partial T}{\partial t}$$

solve for $\frac{\partial T}{\partial t}$:

$$\Rightarrow \boxed{\frac{\partial T}{\partial t} = \frac{(T_0 - T_1)\,\alpha}{D^2}\frac{\partial Y}{\partial t^*}}$$

this can go directly into the starting differential eqn.

Now the right-hand side: $\left(\frac{\partial^2 T}{\partial x^2}\right)$ 5

$$Y = \frac{(T_1 - T)}{(T_1 - T_0)}$$

now take derivative w.r.t. x^*

$$\frac{\partial y}{\partial x^*} = \left(\frac{1}{T_1 - T_0}\right)(-1)\frac{\partial T}{\partial x^*}$$

and again:

$$\boxed{\frac{\partial^2 y}{\partial x^{*2}} = \left(\frac{1}{T_0 - T_1}\right)\underbrace{\frac{\partial^2 T}{\partial x^{*2}}}_{\text{need to relate to } \frac{\partial^2 T}{\partial x^2}}}$$

$$x^* = \frac{x}{D}$$

$$x = D x^*$$

$$\frac{\partial x}{\partial x^*} = D$$

chain rule:

$$\frac{\partial T}{\partial x} \underbrace{\left(\frac{\partial x}{\partial x^*} \right)}_{D} = \frac{\partial T}{\partial x^*}$$

chain rule again, and derivative wrt. x^*:

$$D \frac{\partial T}{\partial x} = \frac{\partial T}{\partial x^*}$$

$$D \frac{\partial T/\partial x}{\partial x} \underbrace{\frac{\partial x}{\partial x^*}}_{D} = \frac{\partial^2 T}{\partial x^{*2}}$$

$$D^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial x^{*2}}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{D^2} \frac{\partial^2 T}{\partial x^{*2}}}$$

Now, pull it all together:

$$\underbrace{\frac{\partial T}{\partial t}}_{\text{From pg 3}} = \alpha \frac{\partial^2 T}{\partial x^2}$$

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$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\left(\frac{\cancel{T_0 - T_1}}{\cancel{D^2}}\right) \cancel{\alpha} \frac{\partial y}{\partial t^*} = \cancel{\alpha} \left(\frac{1}{\cancel{D^2}}\right) \underbrace{\frac{\partial^2 T}{\partial x^{*2}}}_{(\cancel{T_0 - T_1}) \frac{\partial^2 y}{\partial x^{*2}}}$$

from pg 4

$$\boxed{\frac{\partial y}{\partial t^*} = \frac{\partial^2 y}{\partial x^{*2}}}$$

$\rightarrow$ Surprise! No key dimensionless number remains.

$\rightarrow$ But, what about BC & IC ...

IC: $t=0 \quad T=T_0 \quad \forall x$

$$Y = \frac{T-T_0}{T_1-T_0}$$

Initial Condition

$$t=0 \Rightarrow \boxed{\begin{array}{l} t^* = Fo = 0 \\ Y = 0 \\ \forall\ x/D \end{array}}$$

BC: $x = \infty$

$T = T_0$

First B.C.

$$\Rightarrow \boxed{\frac{x}{D} = \infty,\ Y = 0}$$

2nd BC: $x = 0$

$$\underbrace{-k\frac{dT}{dx}}_{?} = h\underbrace{(T_1 - T)}_{=(T_1 - T_0)y} \quad \forall\, t > 0$$

$$\frac{dT}{dx} = \frac{1}{D}\frac{\partial T}{\partial x^*} = \frac{1}{D}\left((T_1 - T_0)(-1)\right)\frac{\partial y}{\partial x^*}$$

substitute back:

$$\frac{\cancel{(T_0 - T_1)}}{D}\frac{\partial y}{\partial x^*}(-k) = h\cancel{(T_1 - T_0)}\,y$$

2nd BC: $\boxed{\dfrac{\partial y}{\partial x^*} = \left(\dfrac{hD}{k}\right) y}$ Biot #