CM3120 Transport/Unit Operations 2

Dimensional Analysis

For Unsteady State Heat Transfer

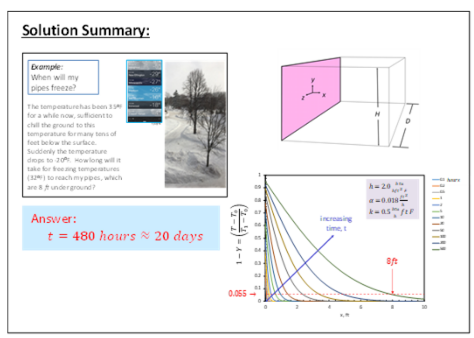
Professor Faith A. Morrison

Department of Chemical Engineering
Michigan Technological University

www.chem.mtu.edu/~fmorriso/cm3120/cm3120.html

 1

We used unsteady state heat transfer modeling to solve one practical problem.



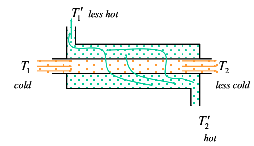
What can we do to extend these methods to a wider class of problems?

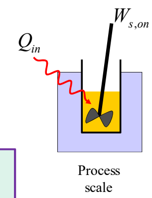
 2

Complex Heat Transfer – Dimensional Analysis

CM3110 REVIEW

(**Answer**: *Use the same techniques we have been using in fluid mechanics)*





Engineering Modeling (complex systems)

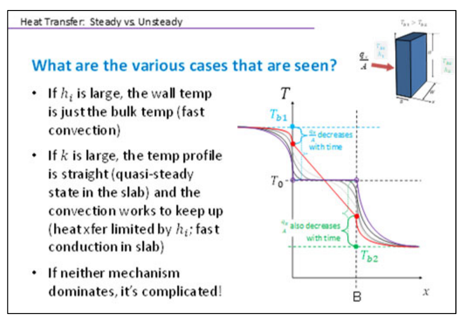
- Choose an idealized problem and solve it
- From insight obtained from **ideal** problem, identify governing equations of **real** problem
- Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
- Design experiments to test modeling thus far
- Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness length scale)
- Design additional experiments
- Iterate until useful correlations result

3

Back to this:

What is our usual strategy for complex phenomena?

Answer: Dimensional Analysis



Engineering Modeling (complex systems)

- ✓ Choose an idealized problem and solve it
- ✓ From insight obtained from **ideal** problem, identify governing equations of **real** problem
- Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
- Design experiments to test modeling thus far
- Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness lengthscale)
- Design additional experiments
- Iterate until useful correlations result

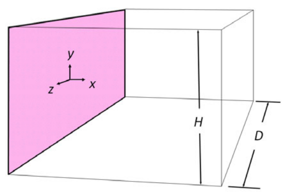
✓ ***Let's nondimensionalize the governing equations and BCs.***

✓ ***Let's sort out the various unsteady cases.***

4

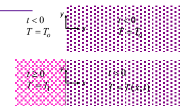
1D Heat Transfer: Unsteady State

Let's nondimensionalize the governing equations and BCs.
Let's sort out the various cases.



1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Initial condition: $t = 0 \quad T = T_0 \quad \forall\ x$

Boundary conditions:

$$x = 0 \quad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \quad t > 0$$

$$x = \infty \quad T = T_0 \quad \forall\ t$$

(Review:
How did we do this before?)

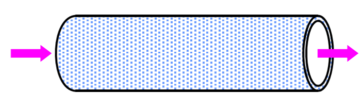
5

CM3110 REVIEW

Method:

- Identify the governing equation(s)
- Choose "typical" values (*scale factors*)
- Use them to scale the equations

We'll modify our solution for
Convective Heat Transfer



Pipe flow

Dimensional Analysis

non-dimensional variables:

time: $t^* \equiv \frac{tV}{D}$

position: $r^* \equiv \frac{r}{D}$, $z^* \equiv \frac{z}{D}$

velocity: $v_z^* \equiv \frac{v_z}{V}$, $v_r^* \equiv \frac{v_r}{V}$, $v_\theta^* \equiv \frac{v_\theta}{V}$

driving force: $P^* \equiv \frac{P}{\rho V^2}$, $g_z^* \equiv \frac{g_z}{g}$

Energy

non-dimensional variables:

position: $r^* \equiv \frac{r}{D}$, $z^* \equiv \frac{z}{D}$

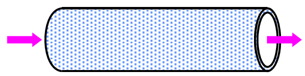
temperature: $T^* \equiv \frac{T - T_o}{(T_1 - T_o)}$

source: $S^* \equiv \frac{S}{S_0}$

6

We'll modify our solution for

Convective Heat Transfer



Pipe flow

Dimensional Analysis

non-dimensional variables:

time: $t^* \equiv \frac{tV}{D}$

position: $r^* \equiv \frac{r}{D}$, $z^* \equiv \frac{z}{D}$

velocity: $v_z^* \equiv \frac{v_z}{V}$, $v_r^* \equiv \frac{v_r}{V}$, $v_\theta^* \equiv \frac{v_\theta}{V}$

driving force: $P^* \equiv \frac{P}{\rho V^2}$, $g_z^* \equiv \frac{g_z}{g}$

$t_{char} = \frac{D}{V}$ (convection)

Energy

non-dimensional variables:

position: $r^* \equiv \frac{r}{D}$, $z^* \equiv \frac{z}{D}$

temperature: $T^* \equiv \frac{T - T_o}{(T_1 - T_o)}$

source: $S^* \equiv \frac{S}{S_0}$

Slight problem: We need to nondimensionalize t for the unsteady case also, but there is no characteristic velocity in thermal conduction in a solid.

7

Choice:

For the unsteady case we'll choose a characteristic time based on the thermal diffusivity, α.

$$t^* \equiv \frac{\alpha t}{D^2}$$

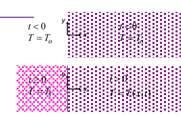
This dimensionless time is called Fourier number Fo.

$$t_{char} = \frac{D^2}{\alpha} = \frac{D^2 \rho \hat{C}_p}{k}$$

(thermal diffusion)

1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Initial condition: $t = 0 \quad T = T_0 \quad \forall\, x$

Boundary conditions:

$x = 0 \quad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \quad t > 0$

$x = \infty \quad T = T_0 \quad \forall\, t$

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

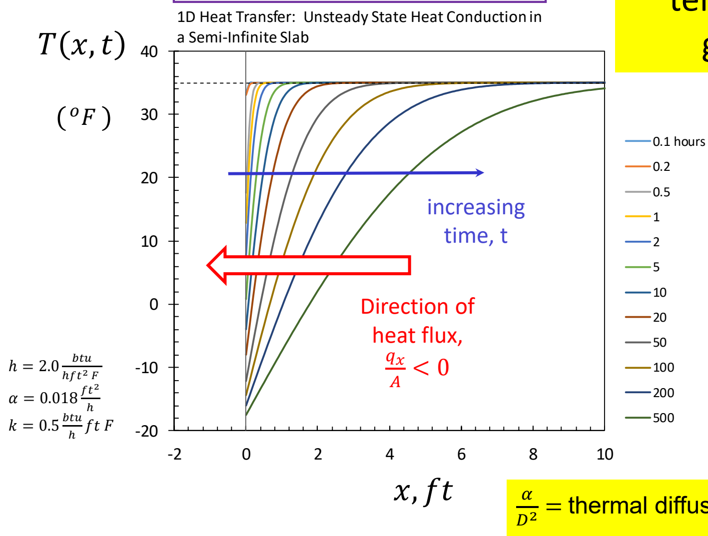
(Appears in the energy balance)

$\frac{D^2}{\alpha}$ = thermal diffusion time = $\frac{D^2 \rho \hat{C}_p}{k}$

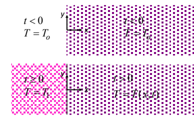
8

1D Heat Transfer: Unsteady State

Example: When will my pipes freeze?



Energy is *diffusing* **down** the temperature gradient



thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

$$\frac{\alpha}{D^2} = \text{thermal diffusion rate} = \frac{k}{D^2 \rho \hat{C}_p}$$

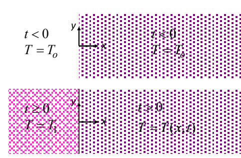
9

Dimensional Analysis, Unsteady State Convection

Non-dimensionalize (eqns, BCs)

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

$$q_x = -k \frac{\partial T}{\partial x} = hA(T_1 - T)$$



non-dimensional variables:

position:

$$x^* \equiv \frac{x}{D}$$

temperature:

$$Y = T^* \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

time:

$$t^* \equiv \frac{\alpha t}{D^2}$$

This dimensionless time is called Fourier number Fo.

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

10

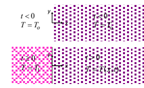
1D Heat Transfer: Unsteady State

Let's do it.

Dimensional Analysis, Unsteady State Convection

Non-dimensionalize (eqns, BCs)

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right) \qquad q_x = -k\frac{\partial T}{\partial x} = hA(T_1 - T)$$



non-dimensional variables:

position: $x^* \equiv \frac{x}{D}$

temperature: $Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$

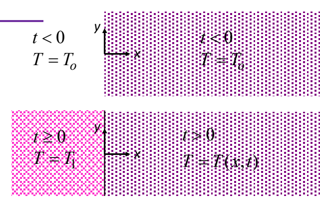
time: $t^* \equiv \frac{\alpha t}{D^2}$

This dimensionless time is called Fourier number Fo.

Fo – Fourier Number $= \frac{\alpha t}{D^2}$

1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial Y}{\partial t^*} = \frac{\partial^2 Y}{\partial x^{*2}}$$

temperature: $Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$

Initial condition: $t^* = 0 \quad Y = 1 \quad \forall\, x^*$

Boundary conditions: $x^* = \infty \quad Y = 1 \quad \forall\, t^*$

$$x^* = 0 \quad \frac{\partial Y}{\partial x^*} = \mathrm{Bi}\, Y \quad t^* > 0$$

$$\mathrm{Bi} \equiv \frac{hD}{k}$$

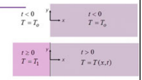
Bi – Biot Number $= \frac{hD}{k}$

In dimensionless form, we see that this problem reduces to

$$Y = Y\left(\frac{x}{D}, \text{Fo}, \text{Bi}\right)$$

1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Initial condition: $t=0 \quad T=T_0 \quad \forall\ x$

Boundary conditions:

$x=0 \quad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \quad t>0$

$x=\infty \quad T=T_0 \quad \forall\ t$

Dimensionless quantities:

$Y = \frac{(T_1 - T)}{(T_1 - T_0)}$ — **Y** (dimensionless temperature interval)

$t^* = \text{Fo} = \frac{\alpha t}{D^2}$ — **Fourier number** (dimensionless time based on thermal diffusion)

$x^* = \frac{x}{D}$

$\text{Bi} = \frac{hD}{k}$ — **Biot number** (pronounced BEE-OH) Ratio of heat transfer resistance at the boundary to resistance in the solid. This is a *transport* issue.

13

Because we can solve this problem analytically, **we can confirm** that the dimensional analysis is correct:

Solution:

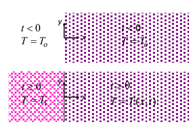
this:

Unsteady State Heat Conduction in a Semi-Infinite Slab

Solution:

$$\frac{T - T_0}{T_1 - T_0} = \text{erfc}\,\zeta - e^{\beta(2\zeta+\beta)}\,\text{erfc}(\zeta + \beta)$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$



$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

+

$$\text{Bi} - \text{Biot Number} = \frac{hD}{k} \qquad \text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

=

$$1 - Y = \text{erfc}\left(\frac{x}{D}\frac{1}{2}\frac{1}{\sqrt{\text{Fo}}}\right) - e^{\text{Bi}\left(\frac{x}{D}\right) + \text{Bi}^2\text{Fo}}\,\text{erfc}\left(\sqrt{\text{Fo}}\left(\text{Bi} + \frac{x}{D}\frac{1}{\text{Fo}}\right)\right)$$

14

Because we can solve this problem analytically, **we can confirm** that the dimensional analysis is correct:

$$1 - Y = \operatorname{erfc}\left(\frac{x}{D}\frac{1}{2}\frac{1}{\sqrt{\text{Fo}}}\right) - e^{\text{Bi}\left(\frac{x}{D}\right)+\text{Bi}^2\text{Fo}}\operatorname{erfc}\left(\sqrt{\text{Fo}}\left(\text{Bi} + \frac{x}{D}\frac{1}{\text{Fo}}\right)\right)$$

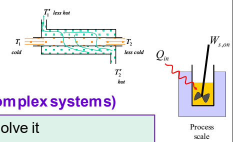
This is like laminar flow, where we can solve the problem all the way to the end and we find $f = \frac{16}{Re}$

And we don't need correlations

The dimensional analysis provides, however, the "**lay of the land**" for these types of problems.

Complex Heat Transfer – Dimensional Analysis

(***Answer***: *Use the same techniques we have been using in fluid mechanics*)



Engineering Modeling (complex systems)

- Choose an idealized problem and solve it
- From insight obtained from **ideal** problem, identify governing equations of **real** problem
- Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
- Design experiments to test modeling thus far
- Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness lengthscale)
- Design additional experiments
- Iterate until useful correlations result

15

Unsteady State Heat Transfer in a Body

Two Additional Dimensionless Numbers

$$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

momentum
energy
mass

Dimensionless Numbers

Re – Reynolds $= \frac{\rho V D}{\mu} = \frac{VD}{\nu}$

Fr – Froude $= \frac{V^2}{gD}$

Pe – Péclet$_h$ = RePr $= \frac{\hat{C}_p \rho V D}{k} = \frac{VD}{\alpha}$

Pe – Péclet$_m$ = ReSc $= \frac{VD}{D_{AB}}$

These numbers from the governing equations tell us about the relative importance of the terms they precede in the microscopic balances (***scenario*** *properties*).

Pr – Prandtl $= \frac{\hat{C}_p \mu}{k} = \frac{\nu}{\alpha}$

Sc – Schmidt = LePr $= \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$

Le – Lewis $= \frac{\alpha}{D_{AB}}$

These numbers compare the magnitudes of the diffusive transport coefficients ν, α, D_{AB} (***material*** *properties*).

f – Friction Factor $= \frac{\mathcal{F}_{drag}}{\left(\frac{1}{2}\rho V^2\right) A_c}$

Nu – Nusselt $= \frac{hD}{k}$

Sh – Sherwood $= \frac{k_m D}{D_{AB}}$

These numbers are defined to help us build transport data correlations based on the fewest number of grouped (dimensionless) variables (***scenario*** *properties*).

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

Scales the time evolution of the temperature profile relative to the material's thermal properties, $\alpha = k/\rho\hat{C}_p$ (thermal diffusion time).

16

Dimensional Analysis in Unsteady State Heat Transfer

Warning!

Note Two Different Numbers

with completely different purposes and meanings
but confusingly similar definitions

$$\text{Bi} - \text{Biot Number} = \frac{hD}{k} = \frac{hD_{\text{body}}}{k_{\text{body}}}$$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h) for a body in contact with a moving fluid.

$$\text{Nu} - \text{Nusselt Number} = \frac{hD}{k} = \frac{hD_{flow}}{k_{fluid}}$$

Dimensionless heat transfer coefficient in convection. Quantifies the physics in the moving fluid and how this results in a resistance to heat transfer, captured in the heat transfer coefficient.

17

Dimensional Analysis in Heat Transfer

Note also:

$$t_{char,1} = \frac{D}{V}$$
(forced convection)

$$t_{char,2} = \frac{D^2}{\alpha} = \frac{D^2 \rho \hat{C}_p}{k}$$
(thermal diffusion)

$$\frac{convective\ rate}{diffusive\ rate} = \frac{1/t_{char,1}}{1/t_{char,2}}$$

$$= \frac{V/D}{k/D^2 \rho \hat{C}_p} = \frac{\rho V D \hat{C}_p}{k} = \text{Pe}$$

momentum
energy
mass

Dimensionless Numbers

$\text{Re} - \text{Reynolds} = \frac{\rho V D}{\mu} = \frac{VD}{\nu}$

$\text{Fr} - \text{Froude} = \frac{V^2}{gD}$

$\text{Pe} - \text{Péclet}_h = \text{RePr} = \frac{\hat{C}_p \rho V D}{k} = \frac{VD}{\alpha}$

$\text{Pe} - \text{Péclet}_m = \text{ReSc} = \frac{VD}{D_{AB}}$

These numbers from the governing equations tell us about the relative importance of the terms they precede in the microscopic balances (***scenario*** *properties*).

$\text{Pr} - \text{Prandtl} = \frac{\hat{C}_p \mu}{k} = \frac{\nu}{\alpha}$

$\text{Sc} - \text{Schmidt} = \text{LePr} = \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$

$\text{Le} - \text{Lewis} = \frac{\alpha}{D_{AB}}$

These numbers compare the magnitudes of the diffusive transport coefficients ν, α, D_{AB} (***material*** *properties*).

$f - \text{Friction Factor} = \frac{\mathcal{F}_{drag}}{\left(\frac{1}{2}\rho V^2\right) A_c}$

$\text{Nu} - \text{Nusselt} = \frac{hD}{k}$

$\text{Sh} - \text{Sherwood} = \frac{k_m D}{D_{AB}}$

These numbers are defined to help us build transport data correlations based on the fewest number of grouped (dimensionless) variables (***scenario*** *properties*).

Non-dimensional Energy Equation

energy

$$\left(\frac{\partial T^*}{\partial t^*} + \underline{v}^* \cdot \nabla^* T^*\right) = \frac{1}{\text{RePr}} \left(\nabla^{*2} T^*\right) + S^*$$

$\text{Pe} - \text{Péclet}_h = \text{RePr}$
$\text{Pr} - \text{Prandtl}$

The Peclet number is the ratio of convective and diffusive heat transport rates

18

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi:
low k,
high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi:
nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi:
high k,
low h

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat tran... the ...

High Bi:

When the temperature is uniform in the body, we can do a macroscopic energy balance to solve many problems of interest. This is called a "lumped parameter analysis."

At m... affe... bod... the surface.

dominates

$T = T(t)$

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi:
high k,
low h

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

$T = T(x, y, z, t)$, easy BC

At high Bi, <u>the surface temperature equals the bulk temperature</u>; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by body and t the surface

Moderate Bi:

When the wall temperature and the bulk temperature are equal, the microscopic energy balance is easier to carry out (temperature boundary conditions).

At low Bi, t uniform in transfer is limited by rate of heat transfer to the surface (h).

high k, low h

21

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

When both processes affect the outcomes, the full solution may be necessary. For uniform starting temperatures, the solutions are published.

the body.

high h

$T = T(x, y, z, t)$

hard BC

At moderate Bi, heat transfer is affected by <u>both</u> conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is <u>uniform in a finite body</u>; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

22

NEXT: Talk about the three cases

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

23