Last time

CM3120 Transport/Unit Operations 2

Unsteady Macroscopic Energy Balance

Professor Faith A. Morrison
Department of Chemical Engineering
Michigan Technological University

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Summary

$$\mathrm{Bi}_{LP} = \frac{hV_{sys}}{kA_{sys}}$$

LP good for: $\mathrm{Bi}_{LP} < 0.1$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat ...

High Bi:

When the temperature is uniform in the body, we can do a macroscopic energy balance to solve many problems of interest. This is called a "lumped parameter analysis."

... the surface.

dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\mathrm{Bi}_{LP} = hV/kA < 0.1$

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2. Negligible Surface Resistance, $\frac{1}{h}$

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

When the wall temperature and the bulk temperature are equal, the microscopic energy balance is easier to carry out (temperature boundary conditions).

At low Bi, ... uniform in ... transfer is limited by rate of heat transfer to the surface (h).

high k, low h

High Bi:
low k,
high h

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

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Negligible Surface Resistance, $\frac{1}{h}$

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High Bi:
low k,
high h

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

$$\lim_{h\to\infty} |T_{bulk} - T_{wall}| = 0$$

$$\Rightarrow T_{wall} = T_{bulk}$$

We have done many examples with constant temperature boundary conditions.

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Unsteady State Heat Transfer

Summary

High Bi: dominated by internal temperature variation ⇒ solve with temperature boundary conditions; $\text{Bi} = hD_{char}/k$ **(D_{char} varies with the problem)**

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by ... body and t... the surface

Moderate Bi:

At low Bi, t... uniform in ... transfer is limited by rate of heat transfer to the surface (h).

high k, low h

When the wall temperature and the bulk temperature are equal, the microscopic energy balance is easier to carry out (temperature boundary conditions).

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\text{Bi}_{LP} = hV/kA < 0.1$

LP good for: $\text{Bi}_{LP} < 0.1$

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3. No Mechanism Dominates

Moderate Bi: nether process dominates

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

When both processes affect the outcomes, the full solution may be necessary; for uniform starting temperatures there are charts.

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

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Low Bi: high k, low h

Moderate Bi:
nether process dominates

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

This is the most complicated set of cases.

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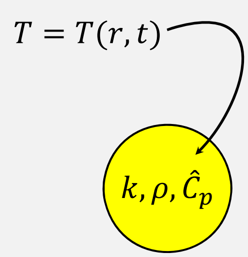
Unsteady State Heat Transfer: Intermediate Biot Number

No Mechanism Dominates

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

- We need to devise an experiment
- Both internal (D/k) and external $(1/h)$ resistances are important
- We need to match *measurable* quantities with *calculable* quantities
- ⇒ ***Microscopic*** Energy Balance
- ⇒*Uncertainty considerations*



Fluid bulk temperature= T_∞

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Unsteady State Heat Transfer: Intermediate Biot Number

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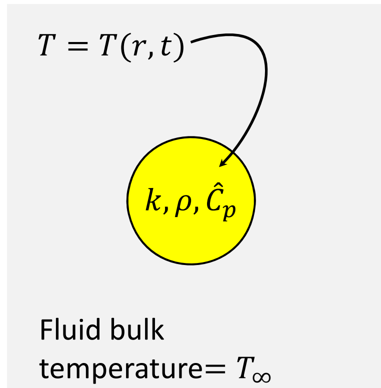
?



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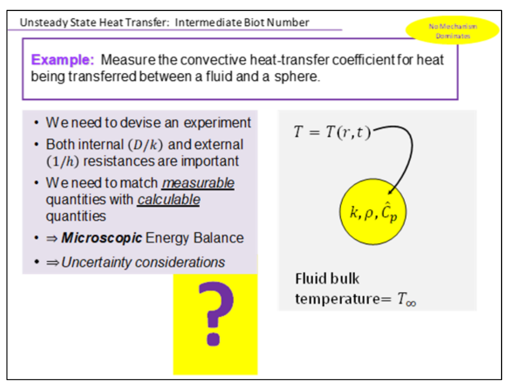

Thinking

- Create an unsteady state heat transfer situation…
- Measure . . .?
- Compare . . .?
- Consider uncertainty in measurements . . . ?

You try.

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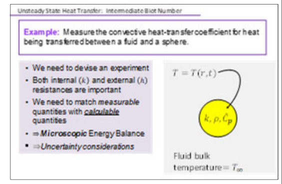
Unsteady State Heat Transfer: Intermediate Biot Number

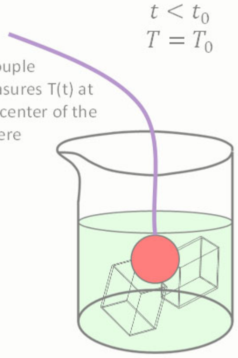


Experiment: Measure $T(t)$ at the center of a sphere $(r = 0)$:

Initially:

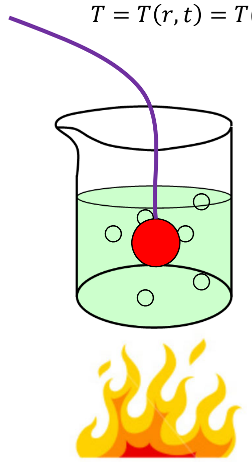
$t < t_0$
$T = T_0$

T-couple measures T(t) at the center of the sphere



Suddenly:

$t \geq t_0$
$T = T(r,t) = T(0,t)$



Unsteady state heat transfer takes place.

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Unsteady State Heat Transfer: Intermediate Biot Number

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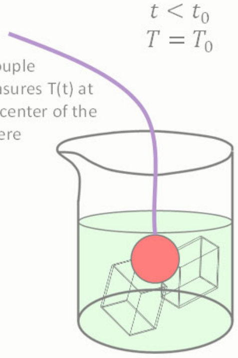

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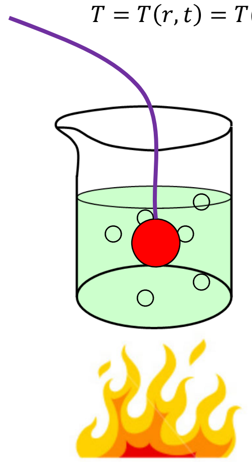

Excel:

$t(s)$	$T(^oC)$
9.50E-02	7.46E+00
2.11E-01	7.44E+00
3.09E-01	7.44E+00
4.09E-01	7.57E+00
5.24E-01	7.46E+00
6.23E-01	7.49E+00
7.39E-01	7.53E+00
8.37E-01	7.46E+00
9.54E-01	7.59E+00
1.05E+00	7.53E+00
1.15E+00	7.58E+00
1.27E+00	7.48E+00
1.37E+00	7.57E+00
…	…

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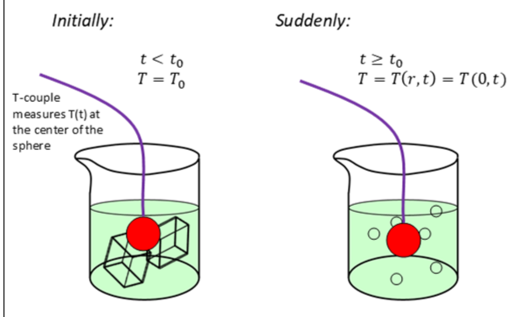
Unsteady State Heat Transfer: Intermediate Biot Number

Modeling

What are the modeling equations?

Experiment: Measure $T(t)$ at the center of a sphere ($r = 0$):



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6.23E-01	7.49E+00
7.39E-01	7.53E+00
8.37E-01	7.46E+00
9.54E-01	7.59E+00
1.05E+00	7.53E+00
1.15E+00	7.58E+00
1.27E+00	7.48E+00
1.37E+00	7.57E+00
...	...

?

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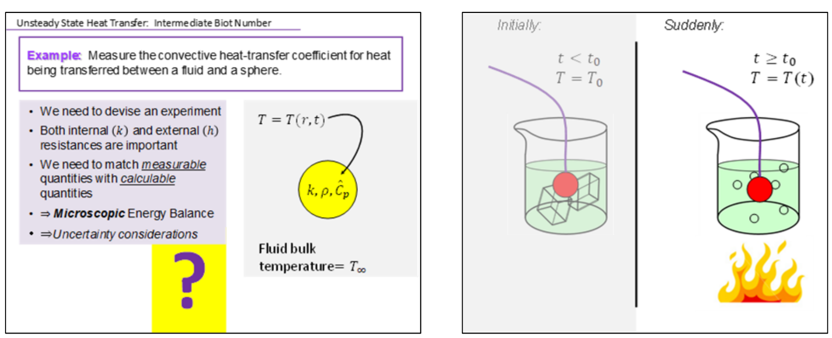

Can we meet our objective?

To determine h:

- Measure center-point temperature as a function of time
- Compare with model predictions, accounting for uncertainty in measurements
- Deduce h

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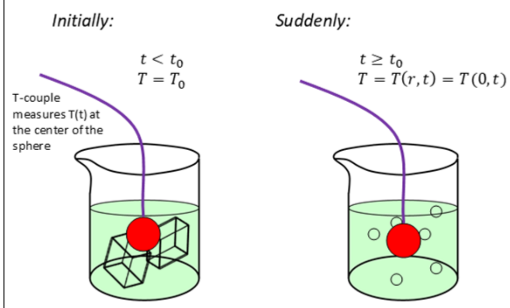
Unsteady State Heat Transfer: Intermediate Biot Number

Modeling

What are the modeling equations?

Experiment: Measure $T(t)$ at the center of a sphere ($r = 0$):



Excel:

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...	...

You try.

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Unsteady State Heat Transfer: Intermediate Biot Number

Microscopic Energy Balance

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right) = k \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S$$

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

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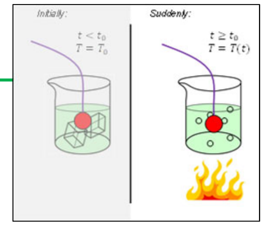
Unsteady State Heat Transfer to a Sphere



Microscopic energy balance in the sphere:

$$\frac{\partial T}{\partial t} = \underbrace{\frac{k}{\rho \hat{C}_p}}_{\equiv \alpha}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right)$$

- Unsteady
- Solid ($\underline{v} = 0$)
- θ, ϕ symmetry
- No current, no rxn

Boundary conditions:

$r = R$, $\quad \frac{q_r}{A} = -k\frac{\partial T}{\partial r} = h(T(r) - T_{bulk}) \quad t > 0$

$r = 0$, $\quad \frac{q_r}{A} = -k\frac{\partial T}{\partial r} = 0 \quad \forall t$

Initial condition:

$t = 0$, $\quad T = T_{initial} \quad \forall r$

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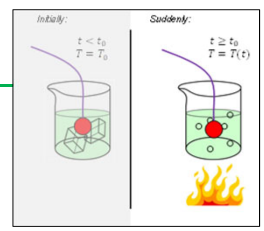
Unsteady State Heat Transfer to a Sphere



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("∀" means "for all")

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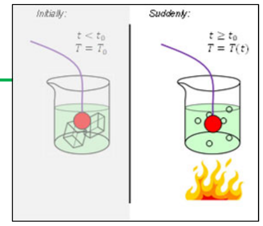
Unsteady State Heat Transfer to a Sphere



Microscopic energy balance in the sphere:

$\equiv \alpha$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)\right)$$

- Unsteady
- Solid ($\underline{v} = 0$)

Bc

Now, Solve

$r = 0,$ $\frac{q_r}{A} = -k\frac{\partial T}{\partial r} = 0$ $\forall t$

Initial condition:

$t = 0,$ $T = T_{initial}$ $\forall r$

("∀" means "for all")

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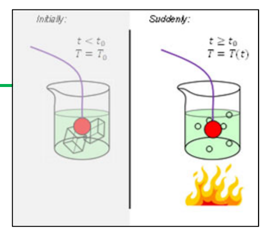
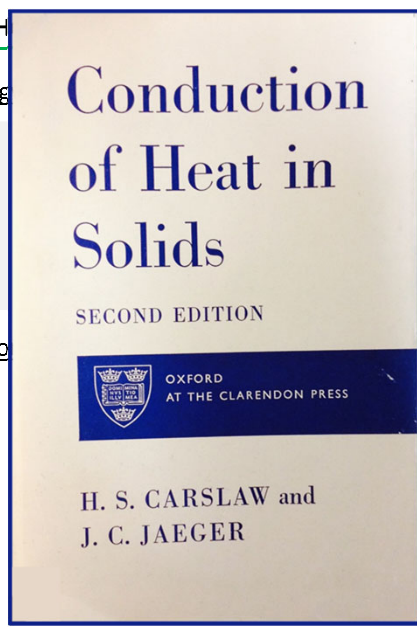
Unsteady State H



Microscopic energ



Unsteady
Solid ($\underline{v} = 0$)
θ, ϕ symmetry
No current, no rxn

Boundary conditio

$r = R,$ $t > 0$

$r = 0,$ $\forall t$

Initial condition:

$t = 0,$ $\forall r$

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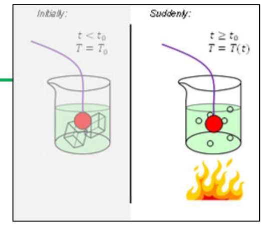
Unsteady State Heat Transfer to a Sphere



Solution:

$$\xi(r,t) \equiv \frac{T(r,t) - T_{bulk}}{T_{initial} - T_{bulk}}$$

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$

Bi = Biot number;
Fo = Fourier number

$$\xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin r\lambda_n}{r\lambda_n}\right)\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Characteristic Equation

(Carslaw and Yeager, 1959, eqn 10, p238)
Incropera and DeWitt, 7th ed, eqn 5.51a, p303

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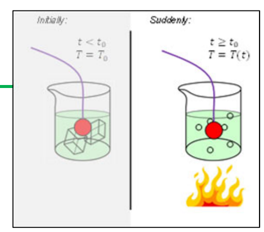
Unsteady State Heat Transfer to a Sphere



Solution:

$$\xi(r,t) \equiv \frac{T(r,t) - T_{bulk}}{T_{initial} - T_{bulk}}$$

Depends on material ($\alpha = k/\rho\hat{C}_p$), and heat transfer processes at surface (h)

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$

Bi = Biot number;
Fo = Fourier number

$$\xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin r\lambda_n}{r\lambda_n}\right)\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Characteristic Equation

We're interested in $T(r,t)$ at the center of the sphere, $r = 0$.

(Carslaw and Yeager, 1959, eqn 10, p238)
Incropera and DeWitt, 7th ed, eqn 5.51a, p303

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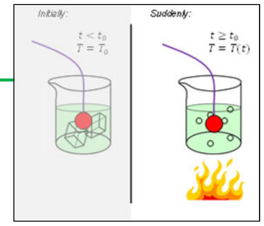
Unsteady State Heat Transfer to a Sphere



What does *this* look like?

$$\xi(0,\mathrm{Fo}) = \frac{T-T_b}{T_i-T_b} = 2\mathrm{Bi}\sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2}\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2+(\mathrm{Bi}-1)^2}{(R\lambda_n)^2+\mathrm{Bi}(\mathrm{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \mathrm{Bi} - 1 = 0$$

Characteristic Equation

Let's plot it to find out. (Excel)

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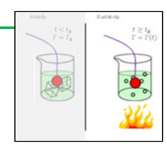
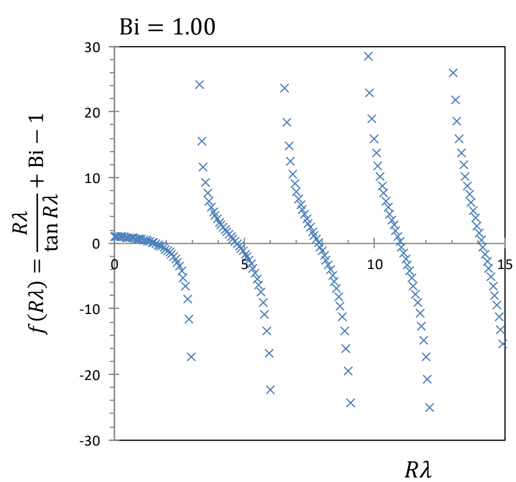
Unsteady State Heat Transfer to a Sphere

Eigenvalues are the roots of the characteristic equation

$$\mathrm{Bi} \equiv \frac{hR}{k}$$





Characteristic Equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \mathrm{Bi} - 1$$

- The λ_n are the roots (zero crossings) of the characteristic equation
- They depend on Biot number Bi

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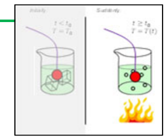
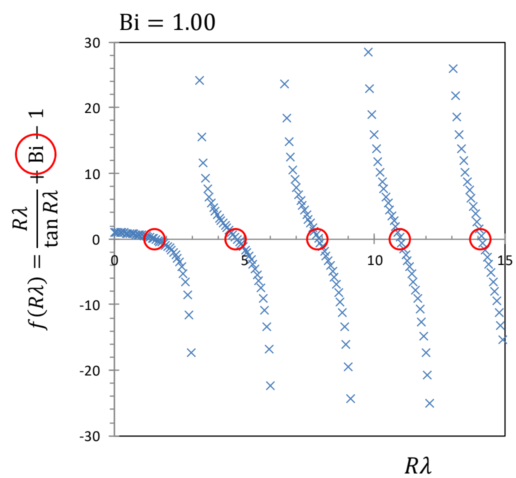
Unsteady State Heat Transfer to a Sphere



$\text{Bi} \equiv \frac{hR}{k}$

Eigenvalues are the roots of the characteristic equation



Characteristic Equation:

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- The λ_n are the roots (zero crossings) of the characteristic equation
- They depend on Biot number Bi

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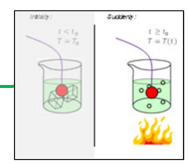
Let's plot it to find out: what are the variables?

$\text{Bi} \equiv \frac{hR}{k}$ $\quad \text{Fo} \equiv \frac{\alpha t}{R^2}$



Solution:

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

Exponential decay with Fo (scaled time)

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-(Fo)(\lambda_n R)^2} \left(\begin{array}{c}\text{bunch of terms} \\ \text{that vary with Bi and } \lambda_n(\text{Bi})\end{array}\right)$$

$\lambda_n(\text{Bi})$ varies only with Bi and n:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

Characteristic Equation

If we choose a fixed Bi, then ξ only varies with Fo

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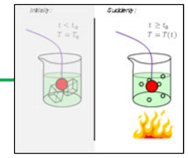
If we choose a **fixed Bi**, then ξ only varies with Fo

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$



For a **fixed Bi:**

$$\xi(0,\text{Fo}) = \frac{T-T_b}{T_i-T_b} = 2\text{Bi}\sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2}\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2+(\text{Bi}-1)^2}{(R\lambda_n)^2+\text{Bi}(\text{Bi}-1)}\right)$$

Exponential decay with Fo (scaled time)

$$\xi(0,\text{Fo}) = \frac{T-T_b}{T_i-T_b} = 2\text{Bi}\sum_{n=1}^{\infty} e^{-(Fo)(\lambda_n R)^2}\left(\begin{matrix}\text{bunch of terms}\\ \text{that vary with Bi and } \lambda_n(\text{Bi})\end{matrix}\right)$$

An infinite sum of decaying **exponentials**

- whose *argument* is Fourier number scaled by something that depends on Biot number and n
- with a prefactor that also depends on Biot number and n

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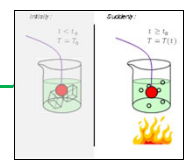
If we choose a **fixed Bi**, then ξ only varies with Fo

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$



For a **fixed Bi:**

$$\xi(0,\text{Fo}) = \sum_{n=1}^{\infty} \tilde{C}_n e^{-\lambda_n^2 R^2 Fo}$$

An infinite sum of decaying **exponentials**

- $\tilde{C}_n$ depends on n through λ_n
- λ_n are calculated (numerically) from the roots of this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Let's plot $\xi(0,\text{Fo})$

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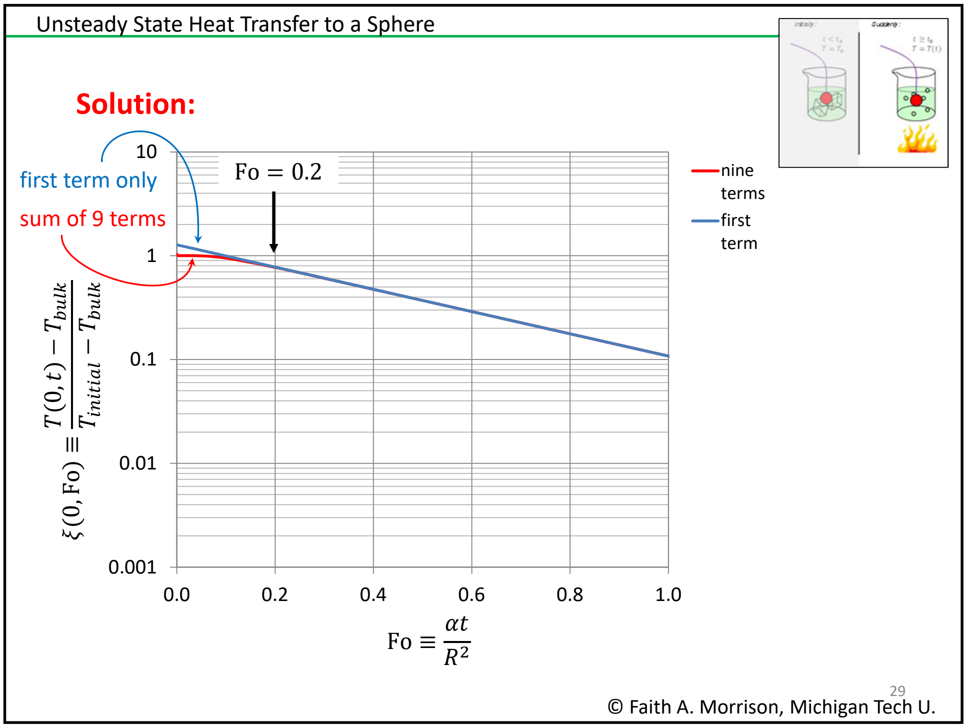
Unsteady State Heat Transfer to a Sphere
Solution:
first term only
sum of 9 terms
Fo = 0.2
10
1
0.1
0.01
0.001
0.0
0.2
0.4
0.6
0.8
1.0
$\xi(0,\mathrm{Fo}) \equiv \frac{T(0,t)-T_{bulk}}{T_{initial}-T_{bulk}}$
$\mathrm{Fo} \equiv \frac{\alpha t}{R^2}$
nine terms
first term
29
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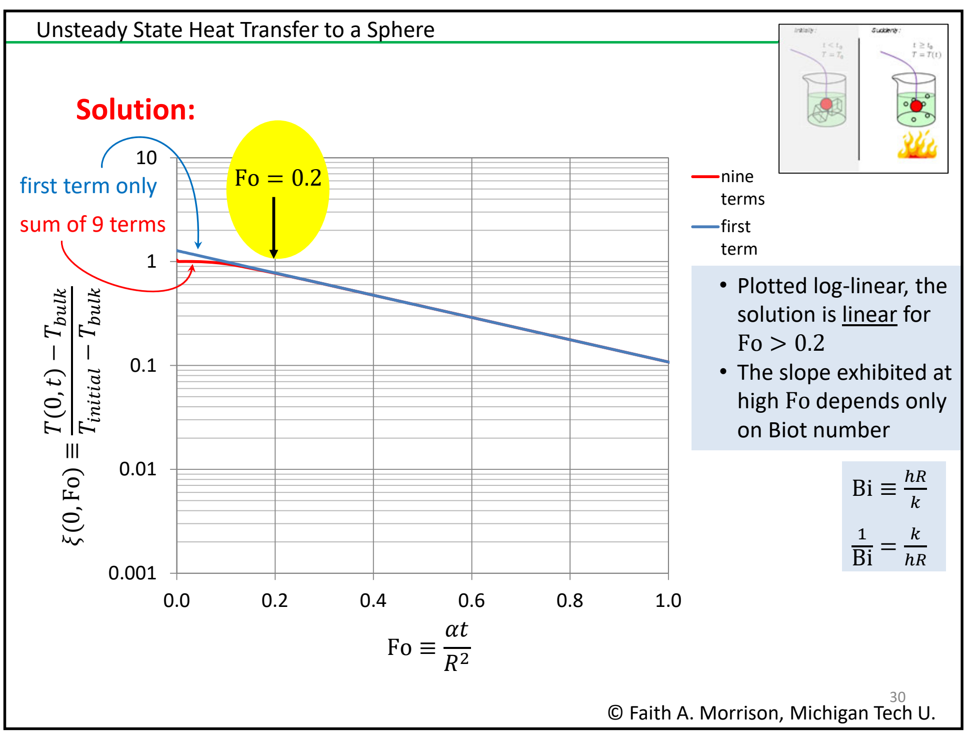
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$\xi(0,\mathrm{Fo}) \equiv \frac{T(0,t)-T_{bulk}}{T_{initial}-T_{bulk}}$
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nine terms
first term
• Plotted log-linear, the solution is linear for $\mathrm{Fo} > 0.2$
• The slope exhibited at high Fo depends only on Biot number
$\mathrm{Bi} \equiv \frac{hR}{k}$
$\frac{1}{\mathrm{Bi}} = \frac{k}{hR}$
30
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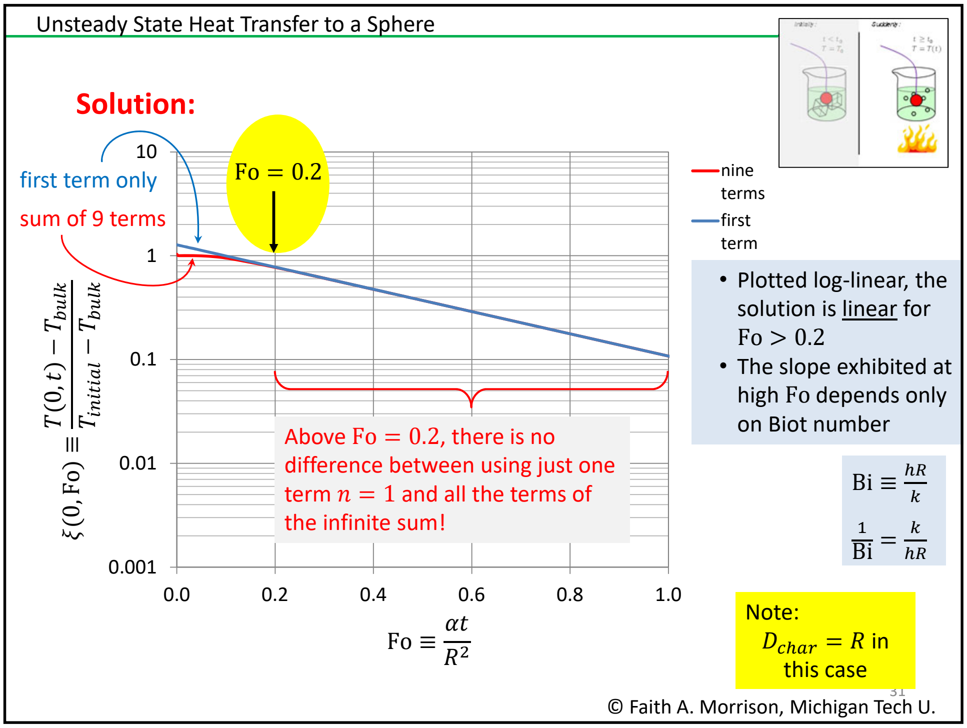
Unsteady State Heat Transfer to a Sphere
Solution:
first term only
sum of 9 terms
Fo = 0.2
nine terms
first term
• Plotted log-linear, the solution is linear for Fo > 0.2
• The slope exhibited at high Fo depends only on Biot number
Above Fo = 0.2, there is no difference between using just one term n = 1 and all the terms of the infinite sum!
$\mathrm{Bi} \equiv \frac{hR}{k}$
$\frac{1}{\mathrm{Bi}} = \frac{k}{hR}$
$\xi(0,\mathrm{Fo}) \equiv \frac{T(0,t) - T_{bulk}}{T_{initial} - T_{bulk}}$
$\mathrm{Fo} \equiv \frac{\alpha t}{R^2}$
Note: $D_{char} = R$ in this case
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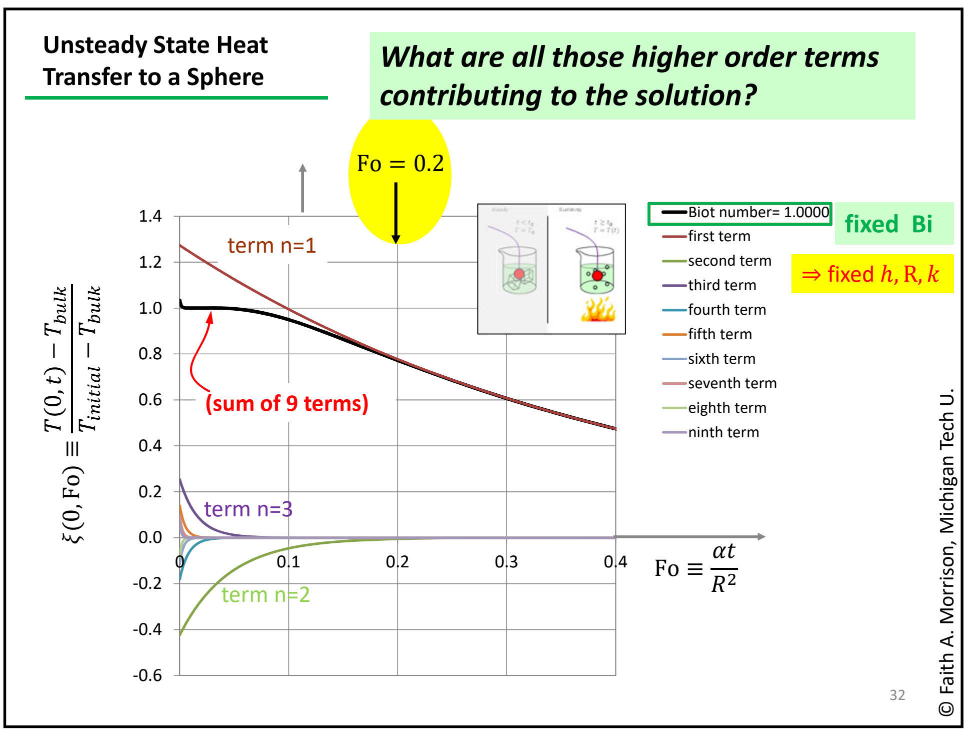
Unsteady State Heat Transfer to a Sphere
What are all those higher order terms contributing to the solution?
Fo = 0.2
term n=1
(sum of 9 terms)
term n=3
term n=2
Biot number= 1.0000
first term
second term
third term
fourth term
fifth term
sixth term
seventh term
eighth term
ninth term
fixed Bi
⇒ fixed h, R, k
$\xi(0,\mathrm{Fo}) \equiv \frac{T(0,t) - T_{bulk}}{T_{initial} - T_{bulk}}$
$\mathrm{Fo} \equiv \frac{\alpha t}{R^2}$
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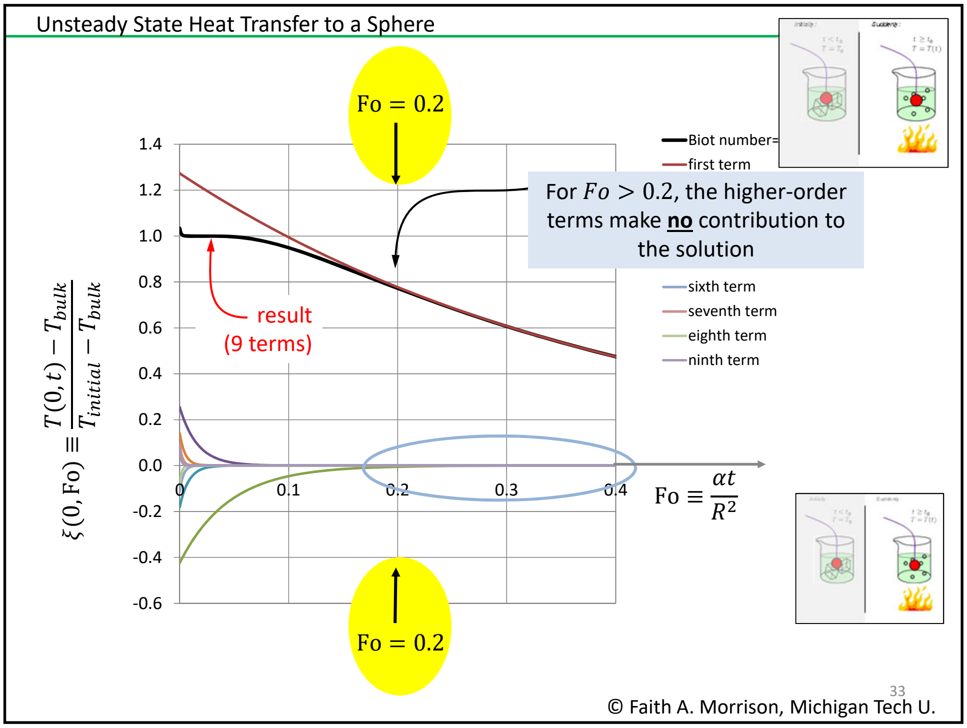
Unsteady State Heat Transfer to a Sphere
Fo = 0.2
Biot number=
first term
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution
sixth term
seventh term
eighth term
ninth term
result
(9 terms)
$\xi(0,\mathrm{Fo}) \equiv \frac{T(0,t) - T_{bulk}}{T_{initial} - T_{bulk}}$
$\mathrm{Fo} \equiv \frac{\alpha t}{R^2}$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
0.1
0.2
0.3
0.4
Fo = 0.2
33
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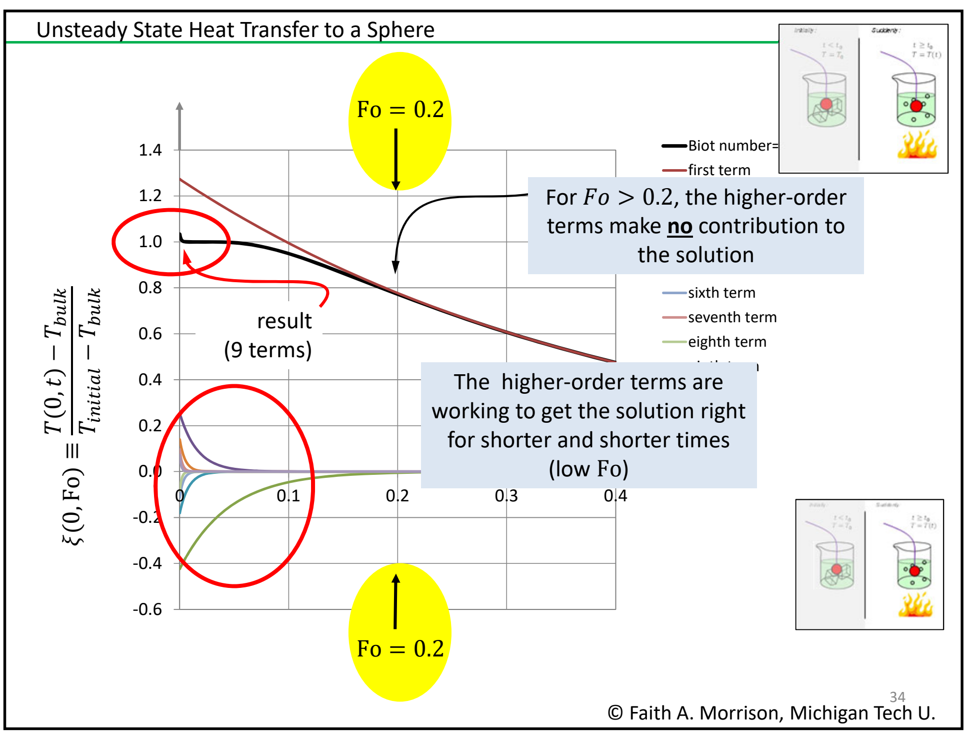
Unsteady State Heat Transfer to a Sphere
Fo = 0.2
Biot number=
first term
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution
sixth term
seventh term
eighth term
result
(9 terms)
The higher-order terms are working to get the solution right for shorter and shorter times (low Fo)
$\xi(0,\mathrm{Fo}) \equiv \frac{T(0,t) - T_{bulk}}{T_{initial} - T_{bulk}}$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
0.1
0.2
0.3
0.4
Fo = 0.2
34
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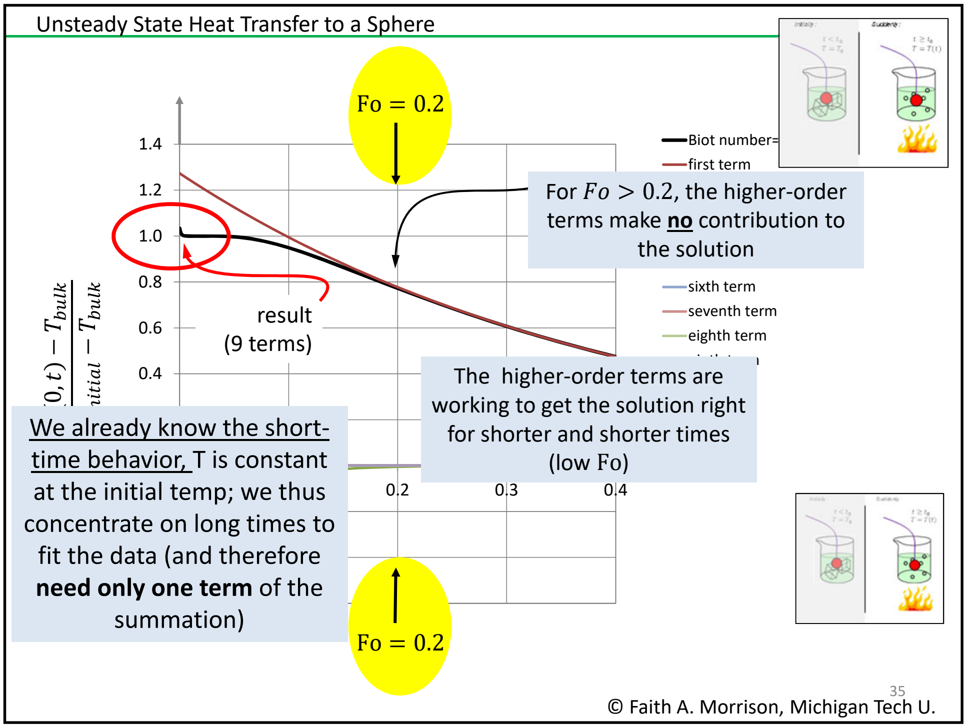
Unsteady State Heat Transfer to a Sphere
Fo = 0.2
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution
Biot number=
first term
sixth term
seventh term
eighth term
1.4
1.2
1.0
0.8
0.6
0.4
result (9 terms)
The higher-order terms are working to get the solution right for shorter and shorter times (low Fo)
We already know the short-time behavior, T is constant at the initial temp; we thus concentrate on long times to fit the data (and therefore **need only one term** of the summation)
0.2
0.3
0.4
Fo = 0.2
35
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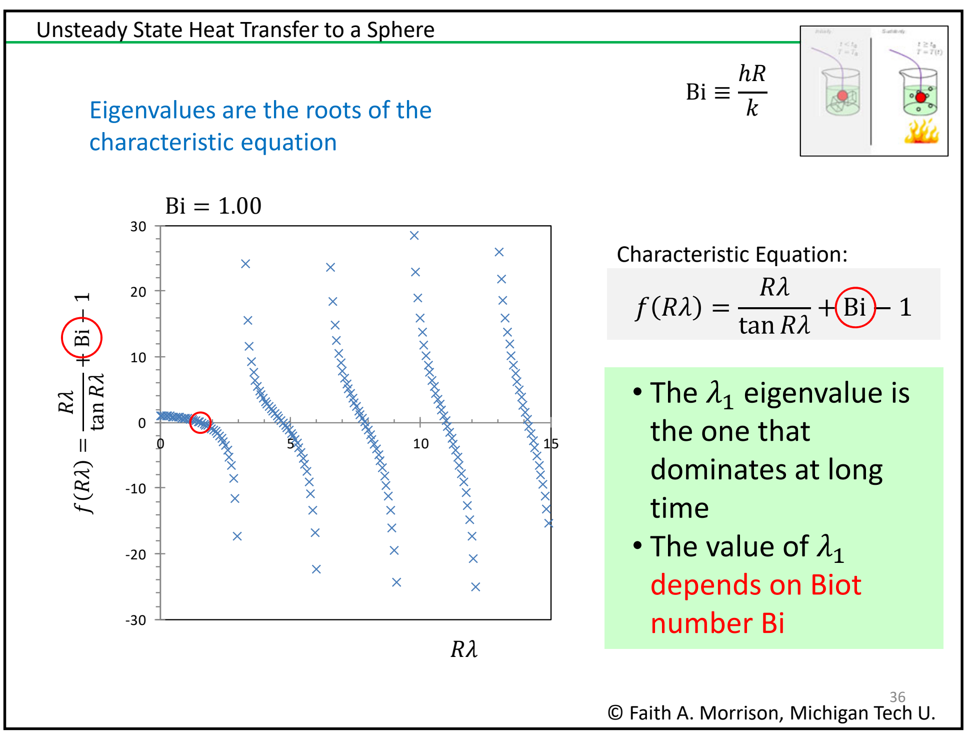
Unsteady State Heat Transfer to a Sphere
$\text{Bi} \equiv \frac{hR}{k}$
Eigenvalues are the roots of the characteristic equation
Bi = 1.00
$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1$
30
20
10
0
-10
-20
-30
0
5
10
15
$R\lambda$
Characteristic Equation:
$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1$
• The λ_1 eigenvalue is the one that dominates at long time
• The value of λ_1 depends on Biot number Bi
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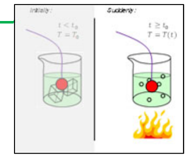
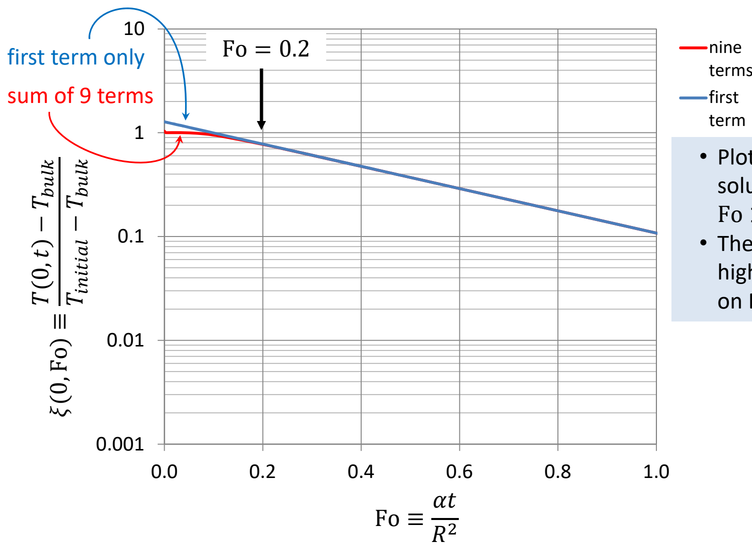
Unsteady State Heat Transfer to a Sphere





- Plotted log-linear, the solution is <u>linear</u> for $\text{Fo} > 0.2$
- The slope exhibited at high Fo depends only on Biot number

$\text{Bi} \equiv \frac{hR}{k}$

$\frac{1}{\text{Bi}} = \frac{k}{hR}$

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Unsteady State Heat Transfer to a Sphere

No Mechanism Dominates

$\text{Bi} \equiv \frac{hR}{k}$

$\text{Fo} \equiv \frac{\alpha t}{R^2}$

Summary

- For a fixed Bi the results are only a function of Fo.
- Solution is an infinite sum of terms.
- Each term corresponds to one eigenvalue, λ_n
- The first term $n = 1$ (λ_1) is the dominant term
- The $n > 1$ terms alternate in sign (positive and negative)
- Higher terms are "fixing" the short time behavior
- At fixed Biot number, the time-dependence is an exponential decay (for $\text{Fo} > 0.2$); this is linear on a log-linear plot versus Fo

So, actually, it turns out all we need are those slopes as a function of Biot number.

<u>Question</u>: How do various values of Biot number affect the heat transfer that occurs?

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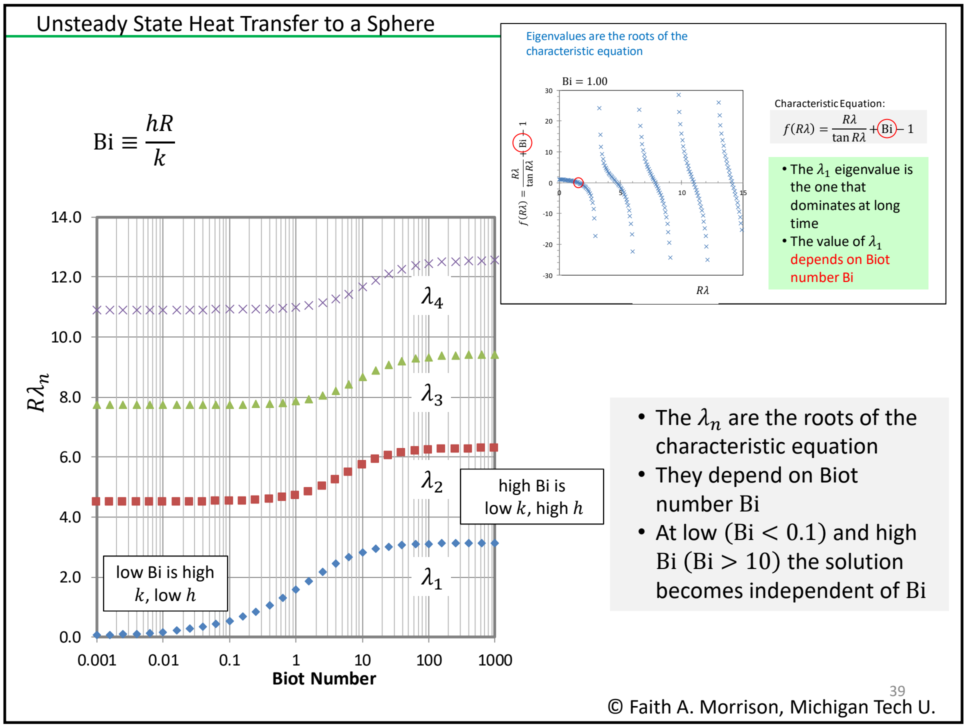
Unsteady State Heat Transfer to a Sphere
$\text{Bi} \equiv \frac{hR}{k}$
Eigenvalues are the roots of the characteristic equation
Bi = 1.00
Characteristic Equation:
$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1$
• The λ_1 eigenvalue is the one that dominates at long time
• The value of λ_1 depends on Biot number Bi
$R\lambda_n$
λ_4
λ_3
λ_2
λ_1
high Bi is low k, high h
low Bi is high k, low h
Biot Number
• The λ_n are the roots of the characteristic equation
• They depend on Biot number Bi
• At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi
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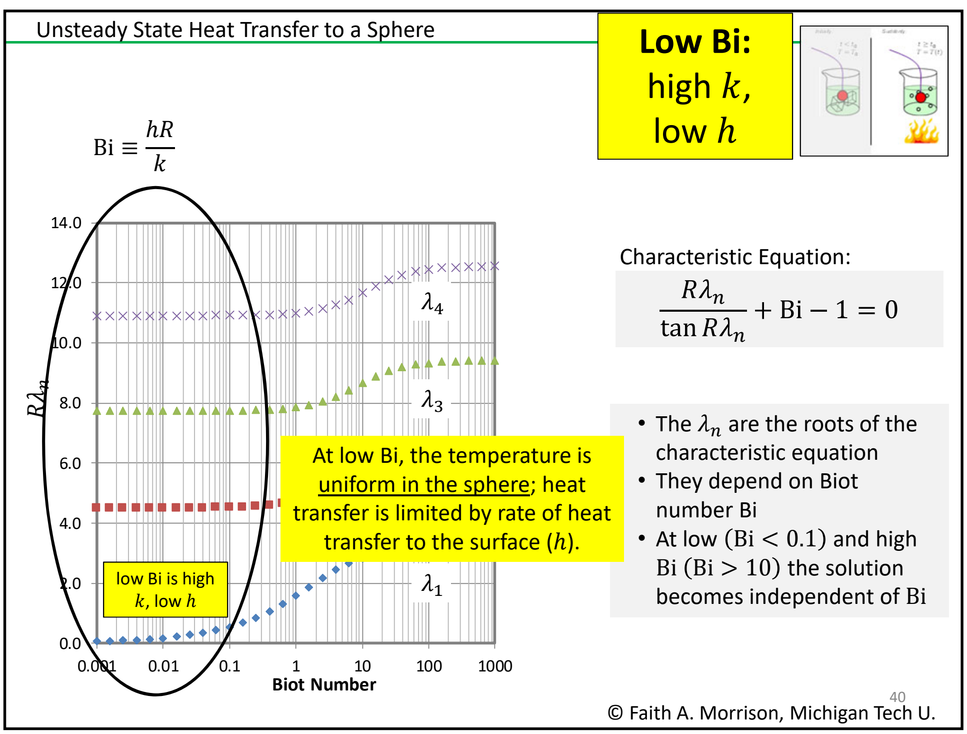
Unsteady State Heat Transfer to a Sphere
Low Bi: high k, low h
$\text{Bi} \equiv \frac{hR}{k}$
Characteristic Equation:
$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$
$R\lambda_n$
λ_4
λ_3
λ_1
At low Bi, the temperature is uniform in the sphere; heat transfer is limited by rate of heat transfer to the surface (h).
low Bi is high k, low h
Biot Number
• The λ_n are the roots of the characteristic equation
• They depend on Biot number Bi
• At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi
40
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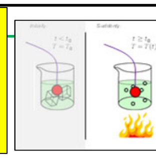
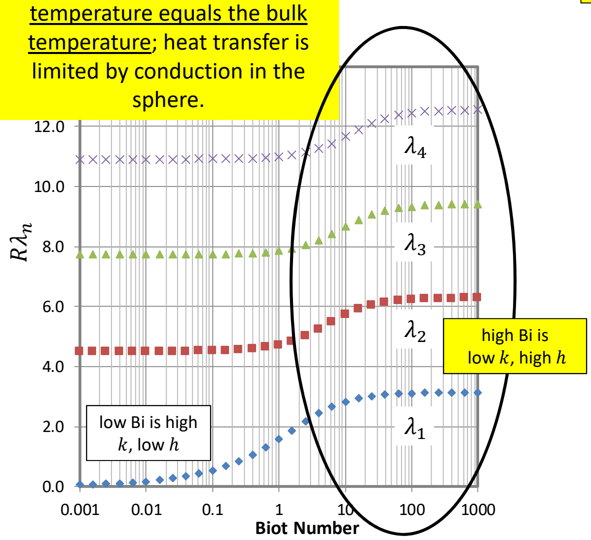
Unsteady State Heat Transfer to a Sphere

High Bi: low k, high h



At high Bi, <u>the surface temperature equals the bulk temperature</u>; heat transfer is limited by conduction in the sphere.



Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

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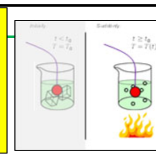
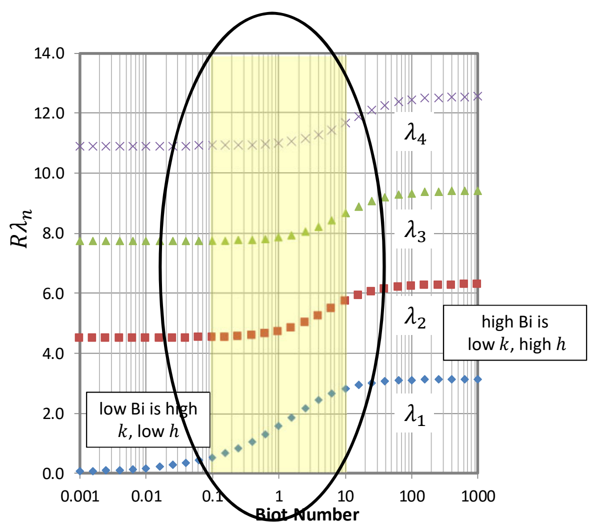
Unsteady State Heat Transfer to a Sphere

Moderate Bi: nether process dominates



At moderate Bi, heat transfer is affected by <u>both</u> conduction in the sphere and the rate of heat transfer to the surface.



Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

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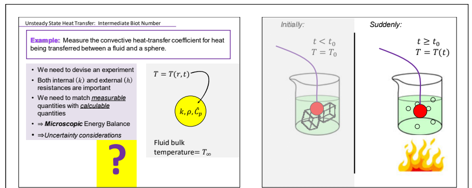
What were we trying to do?

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

Where are we in the process?

✓ We have the model

- We need the measured center-point temperature as a function of time
- We need to compare the two to deduce h.



Can we meet our objective?

To determine h:

- Measure center-point temperature as a function of time
- Compare with model predictions, accounting for uncertainty in measurements
- Deduce h

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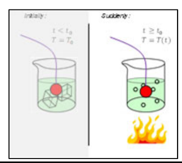
The solution of the model:

$$\xi(0,\mathrm{Fo}) = \sum_{n=1}^{\infty} \tilde{C}_n e^{-\lambda_n^2 R^2 \mathrm{Fo}}$$

$$\mathrm{Fo} \equiv \frac{\alpha t}{R^2} \qquad \mathrm{Bi} \equiv \frac{hR}{k}$$



Use to interpret data.

For a **fixed Bi, Fo > 0.2:**

$$\xi(0,\mathrm{Fo}) \approx \tilde{C}_1 e^{-\lambda_1^2 R^2 \mathrm{Fo}}$$

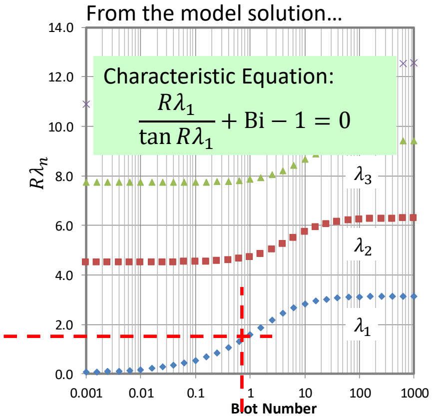
$$\ln \xi(0,\mathrm{Fo}) = \ln(\tilde{C}_1) - \lambda_1^2 R^2 \mathrm{Fo}$$

From experiments...

Plot: $\ln \xi = \ln\left(\frac{T - T_b}{T_i - T_b}\right)$ vs Fo

=> slope $= -\lambda_1^2 R^2$

From the model solution...



Once we know Bi, we can calculate h from Bi

$$\mathrm{Bi} \equiv \frac{hR}{k}$$

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Experimental Data

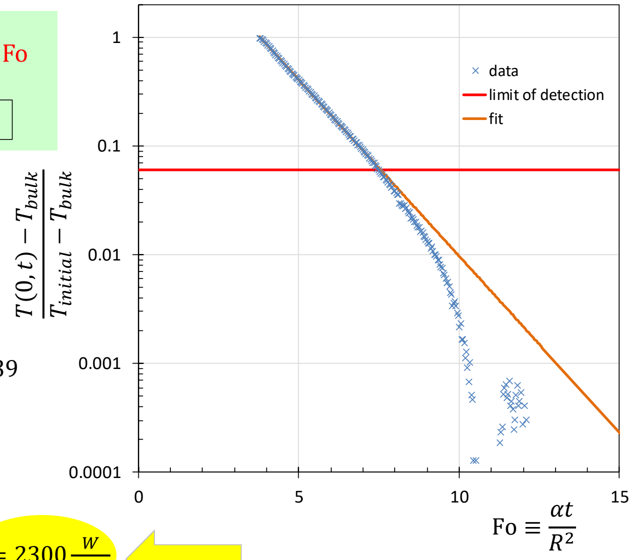
$k = 109\,\frac{W}{mK}$

$R = 0.0127\ m$

From experiments...

Plot: $\ln\xi = \ln\left(\frac{T - T_b}{T_i - T_b}\right)$ vs Fo

=> $\text{slope} = -\lambda_1^2 R^2$



Slope$= -0.74656$

$\lambda_1 R = \sqrt{0.74656} = 0.864039$

Characteristic Equation:

$$\frac{R\lambda_1}{\tan R\lambda_1} + \text{Bi} - 1 = 0$$

$\Rightarrow \text{Bi} = \frac{hR}{k} = 0.2621919,\ h = 2300\,\frac{W}{m^2K}$

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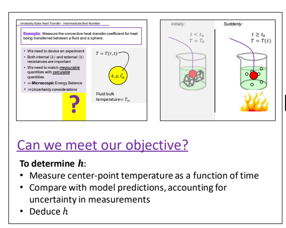
Recap:

Measure convective heat-transfer coefficient h for heat being transferred between a fluid and a sphere.

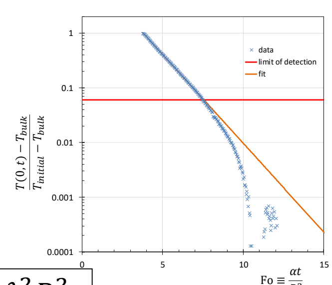
What was the process?

1. Create the scenario and the model of the scenario



2. Take data of center-point $T(t)$

3. Plot the data in a way that we can match it to the model to deduce Bi



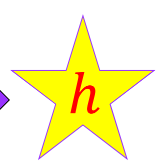
$\text{slope} = -\lambda_1^2 R^2$

Characteristic Equation:

$$\frac{R\lambda_1}{\tan R\lambda_1} + \text{Bi} - 1 = 0$$

4. Calculate h.

$\text{Bi} \equiv \frac{hR}{k}$



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Unsteady State Heat Transfer

Summary

High Bi: dominated by internal temperature variation ⇒ solve with temperature boundary conditions; $\mathrm{Bi} = hD_{char}/k$ (D_{char} varies with the problem)

Moderate Bi: The limits for "moderate" are $0.1 \leq \mathrm{Bi} \leq 10$. When Bi is in this range, a more complete solution may be necessary; $\mathrm{Bi} = hD_{char}/k$. ($D_{char}$ varies with the problem)

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\mathrm{Bi} = hV/kA < 0.1$

$\mathrm{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

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Unsteady State Heat Transfer

Summary

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$\mathrm{Bi} - \text{Biot Number} = \frac{hD}{k}$

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Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

D_{char}=characteristic length scale

We use $D_{LP} = V/A$ only for the lumped parameter analysis. We use different D_{char} in other cases.

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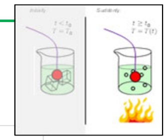
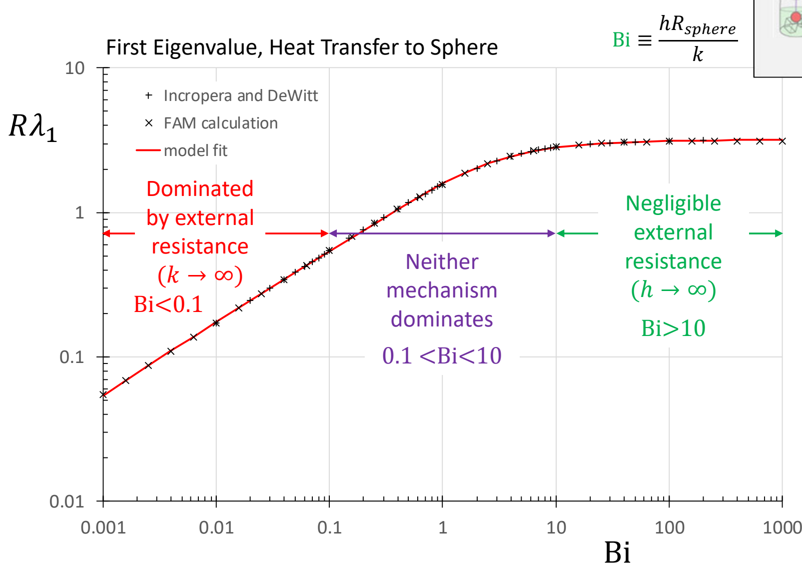
Unsteady State Heat Transfer to a Sphere





Fit by Faith A. Morrison 31 Jan 2019

2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

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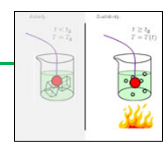
Unsteady State Heat Transfer to a Sphere



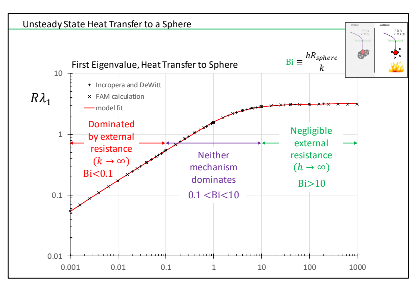
If we know $R\lambda_n$ and we're determining Bi (i.e. h), we use the characteristic equation directly.

If we know h (and hence, Bi) we need to find $\lambda_1 R$ from an iterative solution of the characteristic equation.

Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

Or use a table or correlation for the calculated roots.



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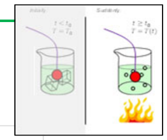
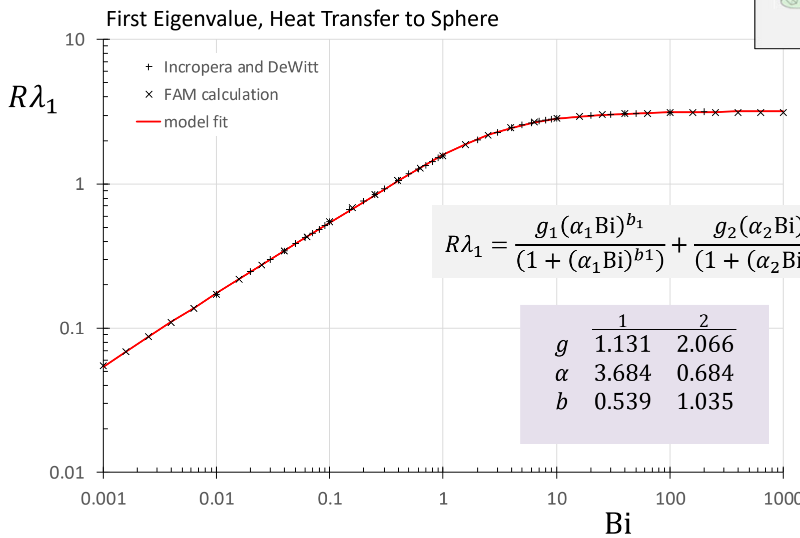
Unsteady State Heat Transfer to a Sphere





$$R\lambda_1 = \frac{g_1(\alpha_1 \mathrm{Bi})^{b_1}}{(1+(\alpha_1 \mathrm{Bi})^{b1})} + \frac{g_2(\alpha_2 \mathrm{Bi})^{b_2}}{(1+(\alpha_2 \mathrm{Bi})^{b2})}$$

	1	2
g	1.131	2.066
α	3.684	0.684
b	0.539	1.035

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2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

(matches true within 1.2%)

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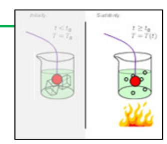
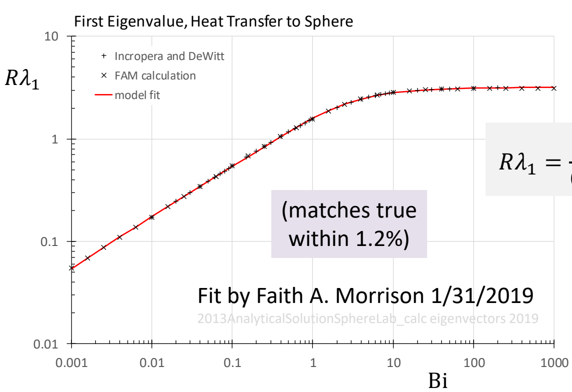
Unsteady State Heat Transfer to a Sphere





(matches true within 1.2%)

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2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

$$R\lambda_1 = \frac{g_1(\alpha_1 \mathrm{Bi})^{b_1}}{(1+(\alpha_1 \mathrm{Bi})^{b1})} + \frac{g_2(\alpha_2 \mathrm{Bi})^{b_2}}{(1+(\alpha_2 \mathrm{Bi})^{b2})}$$

	1	2
g	1.131	2.066
α	3.684	0.684
b	0.539	1.035

For a **fixed Bi**, $\mathrm{Fo} > \mathbf{0.2}$:

$$\xi(0, \mathrm{Fo}) \approx \tilde{C}_1 e^{-\lambda_1^2 R^2 Fo}$$

$$\frac{2\mathrm{Bi}[(R\lambda_1)^2 + (\mathrm{Bi}-1)^2]}{(R\lambda_1)^2 + \mathrm{Bi}(\mathrm{Bi}-1)}\left(\frac{\sin R\lambda_1}{R\lambda_1}\right) = \tilde{C}_1 = \frac{4[\sin(\lambda_1 R) - \lambda_1 R \cos(\lambda_1 R)]}{2\lambda_1 R - \sin 2\lambda_1 R}$$

The two expressions may be shown to be equivalent by substituting the characteristic equation; see FAM notebook v.53 2019; the second version is more robust due to avoiding multiplying by zero when $R\lambda_1 \to \pi$.

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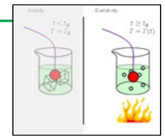
Unsteady State Heat Transfer to a Sphere



Reasonable estimates of the sphere (slab, cylinder) solutions may also be obtained from the "Heisler Charts"

Heisler charts

See:

Geankoplis

Wikipedia

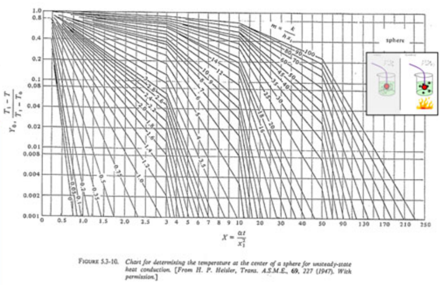
Welty, Rorrer, and Foster (appendix F)

D_{char}=characteristic length scale

For the Heisler chart for spheres, we use $D_{char} = R$. Note this is not the same as what is used in lumped parameter analysis

$$\text{Bi} = \frac{hD_{char}}{k}$$

$D_{char} = x_1 =$ R (sphere)
R (cylinder)
B (slab of thickness $2B$)



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Literature solutions to Unsteady State Heat Transfer to a Sphere

Heisler charts (Geankoplis; see also Wikipedia)

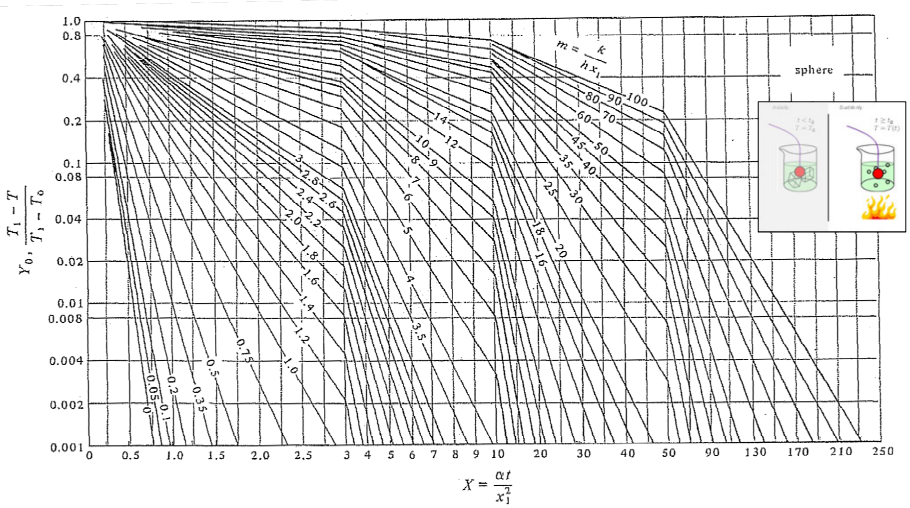

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction.* [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]

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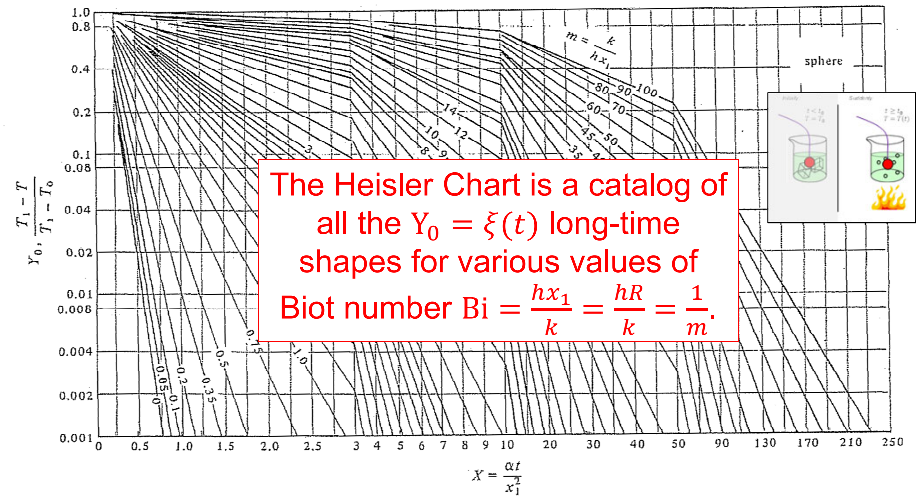
Literature solutions to Unsteady State Heat Transfer to a Sphere

Heisler charts (Geankoplis; see also Wikipedia)



The Heisler Chart is a catalog of all the $Y_0 = \xi(t)$ long-time shapes for various values of Biot number $\mathrm{Bi} = \frac{hx_1}{k} = \frac{hR}{k} = \frac{1}{m}$.

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]* 55

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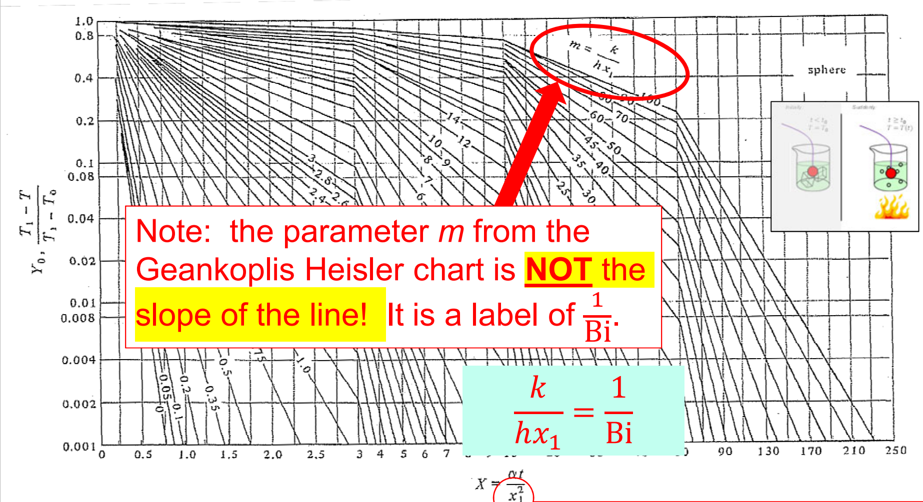
Literature solutions to Unsteady State Heat Transfer to a Sphere

Heisler charts (Geankoplis; see also Wikipedia)



Note: the parameter *m* from the Geankoplis Heisler chart is **NOT** the slope of the line! It is a label of $\frac{1}{\mathrm{Bi}}$.

$$\frac{k}{hx_1} = \frac{1}{\mathrm{Bi}}$$

Also, x_1 is the sphere radius

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]* 56

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Heisler charts (Geankoplis; see also Wikipedia)

Note also: think of it as 4 separate graphs

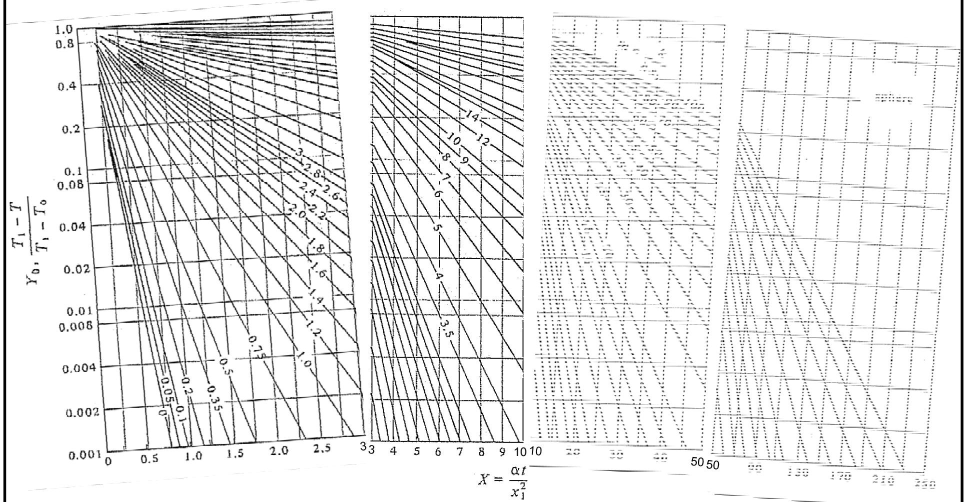

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction.* [*From H. P. Heisler, Trans. A.S.M.E.,* 69, 227 (1947). *With permission.*]

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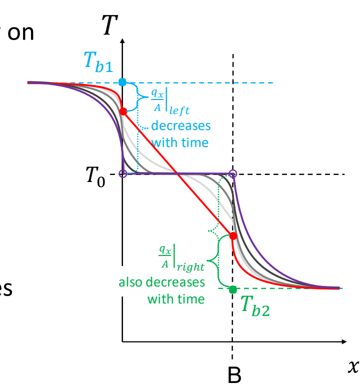
Unsteady State Heat Transfer

Summary

- Unsteady state heat transfer is very common in the chemical process industries
- Temperature distributions depend strongly on what initiates the heat transfer (usually something at the boundary)
- ***Internal resistance*** (D/k) can be limiting, irrelevant, or one among many resistances
- ***External resistance*** $(1/h)$ can be limiting irrelevant, or one among many resistances
- **Dimensional analysis**, once again, organizes the impacts of various influences $(\mathrm{Bi}, \mathrm{Fo})$



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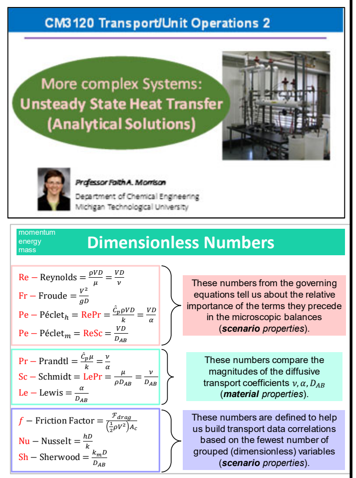
Unsteady State Heat Transfer



Summary *(continued)*

- In transport phenomena, we have dimensionless numbers that represent three important aspects of situations that interest us:
  1. The relative importance of individual terms in the equations of change
  2. The relative magnitudes of the diffusive transport coefficients ν, α, D_{AB}
  3. Scaled values of quantities of interest, e.g. wall forces, heat transfer coefficients, and mass transfer coefficients (data correlations)

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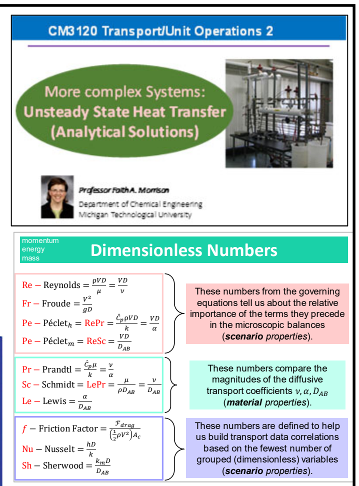
Unsteady State Heat Transfer

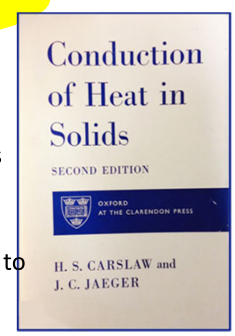


Summary *(continued)*

- If we can develop a model situation for questions of interest, the solutions of the models are often in the literature

Our responsibility in 21st century:

- Learn to develop models that will allow us to estimate or determine answers to the questions that interest us
- Learn to use published solutions (tables, charts) to answer questions that interest us



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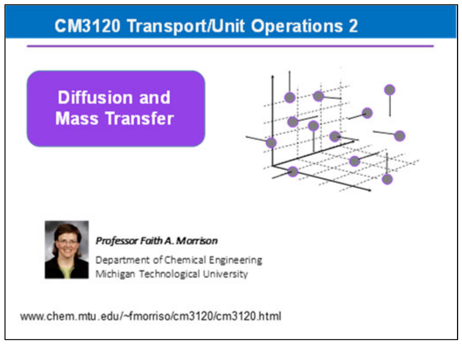
NEXT: Diffusion and Mass Transfer



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