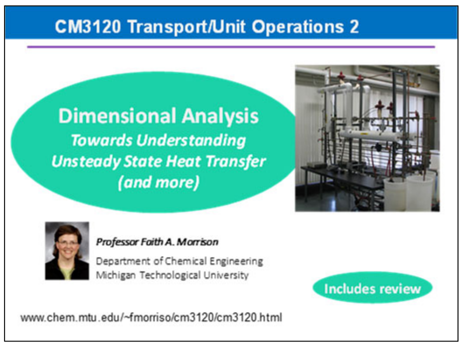
Last time,
Dimensional Analysis for Heat Transfer



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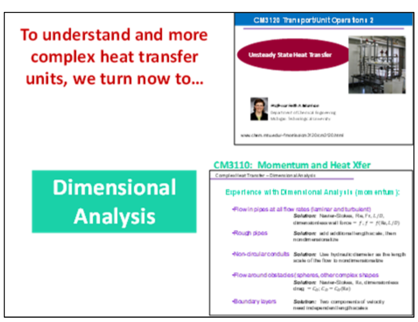

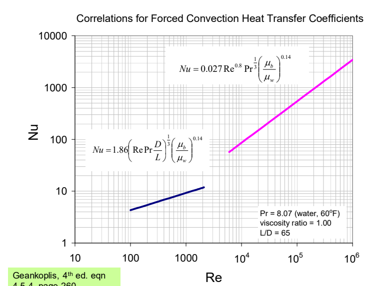

$$\mathrm{Nu} = \mathrm{Nu}\left(\mathrm{Re}, \mathrm{Pr}, \frac{L}{D}\right)$$

Summary

- Dimensional analysis works as well in heat transfer as in momentum transfer
- We should use it (and probably also in mass transfer, but…)
- These dimensionless numbers are stacking up (and…)
- What do they really mean?

2

Dimensional Analysis

These numbers tell us about the relative importance of the terms they precede.

Dimensionless numbers from the **Equations of Change** *(microscopic balances)*

momentum

Non-dimensional Navier-Stokes Equation

$$\left(\frac{\partial v_z^*}{\partial t^*} + \underline{v}^* \cdot \nabla^* v_z^*\right) = -\frac{\partial P^*}{\partial z^*} + \frac{1}{\mathrm{Re}}\left(\nabla^{*2} v_z^*\right) + \frac{1}{\mathrm{Fr}} g^*$$

Re – Reynolds
Fr – Froude

energy

Non-dimensional Energy Equation

$$\left(\frac{\partial T^*}{\partial t^*} + \underline{v}^* \cdot \nabla^* T^*\right) = \frac{1}{\mathrm{RePr}}\left(\nabla^{*2} T^*\right) + S^*$$

Pe – Péclet$_h$ = RePr
Pr – Prandtl

mass

Non-dimensional Continuity Equation (species A)

$$\left(\frac{\partial x_A^*}{\partial t^*} + \underline{v}^* \cdot \nabla^* x_A^*\right) = \frac{1}{\mathrm{ReSc}}\left(\nabla^{*2} x_A^*\right)$$

Pe – Péclet$_m$ = ReSc
Sc – Schmidt

ref: BSL1, p581, 644

3

Dimensionless Numbers

Dimensionless numbers from the **Equations of Change**

Re – Reynolds $= \frac{\rho V D}{\mu} = \frac{VD}{\nu}$

Fr – Froude $= \frac{V^2}{gD}$

Pe – Péclet$_h$ = RePr $= \frac{\hat{C}_p \rho V D}{k} = \frac{VD}{\alpha}$

Pe – Péclet$_m$ = ReSc $= \frac{VD}{D_{AB}}$

These numbers tell us about the **relative importance of the terms** they precede in the microscopic balances (*scenario properties*).

Pr – Prandtl $= \frac{\hat{C}_p \mu}{k} = \frac{\nu}{\alpha}$

Sc – Schmidt = LePr $= \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$

Le – Lewis $= \frac{\alpha}{D_{AB}}$

These numbers compare the **magnitudes of the diffusive transport coefficients** ν, α, D_{AB} (*material properties*).

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

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Dimensional Analysis

These numbers are defined to help us build **transport data correlations** based on the fewest number of grouped (dimensionless) variables (*scenario property*).

Dimensionless numbers from the **Engineering Quantities of Interest**

momentum

Dimensionless Force on the Wall (Drag)

$$f = \frac{1}{\pi}\frac{D}{L}\frac{1}{\text{Re}}\int_0^{\frac{L}{D}}\int_0^{2\pi}\left(\frac{\partial v_z^*}{\partial r^*}\right)\bigg|_{r^*=\frac{1}{2}} d\theta dz^*$$

(Fanning)
f — Friction Factor
$\frac{L}{D}$ — Aspect Ratio

$$f = \frac{\mathcal{F}_{drag}}{\left(\frac{1}{2}\rho V^2\right) A_c}$$

energy

Newton's Law of Cooling

$$Nu = \frac{1}{2\pi L/D}\int_0^{2\pi}\int_0^{L/D} -\frac{\partial T^*}{\partial r^*}\bigg|_{r^*=1/2} dz^* d\theta$$

Nu — Nusselt
$\frac{L}{D}$ — Aspect Ratio

$$\text{Nu} = \frac{hD}{k}$$

mass xfer

Dimensionless Mass Transfer Coefficient

$$Sh = \frac{1}{2\pi}\frac{D}{L}\int_0^{\frac{L}{D}}\int_0^{2\pi}\left(-\frac{\partial x_A^*}{\partial r^*}\right)\bigg|_{r^*=\frac{1}{2}} d\theta dz^*$$

Sh — Sherwood
$\frac{L}{D}$ — Aspect Ratio

$$\text{Sh} = \frac{k_m D}{D_{AB}}$$

5

momentum
energy
mass

Dimensionless Numbers

Re — Reynolds $= \frac{\rho V D}{\mu} = \frac{VD}{\nu}$

Fr — Froude $= \frac{V^2}{gD}$

Pe — Péclet$_h$ = RePr $= \frac{\hat{C}_p \rho V D}{k} = \frac{VD}{\alpha}$

Pe — Péclet$_m$ = ReSc $= \frac{VD}{D_{AB}}$

These numbers from the governing equations tell us about the relative importance of the terms they precede in the microscopic balances (***scenario*** *properties*).

Pr — Prandtl $= \frac{\hat{C}_p \mu}{k} = \frac{\nu}{\alpha}$

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These numbers compare the magnitudes of the diffusive transport coefficients ν, α, D_{AB} (***material*** *properties*).

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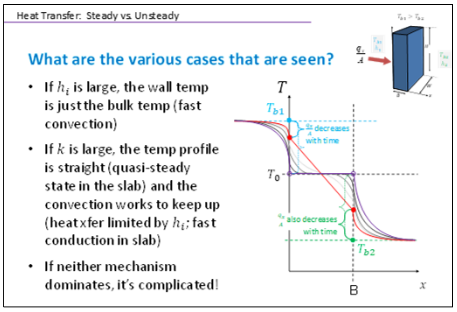
Unsteady State Heat Transfer: Dimensional Analysis

NEW STUFF!

***Question*: What now?**

Answer: Let's apply Dimensional Analysis to something new, unsteady state heat transfer, to sort out the various effects.



Engineering Modeling (complex systems)

- Choose an idealized problem and solve it
- From insight obtained from **ideal** problem, identify governing equations of **real** problem
- Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
- Design experiments to test modeling thus far
- Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness lengthscale)
- Design additional experiments
- Iterate until useful correlations result

7

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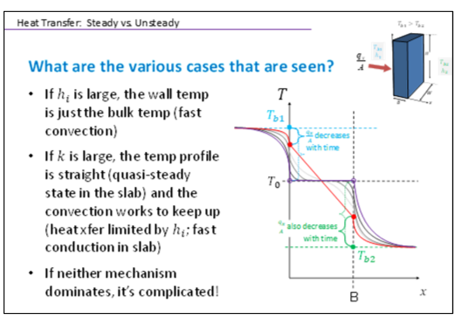

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SPOILER ALERT: There'll be some new dimensionless numbers!

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CM3120 Transport/Unit Operations 2

More complex Systems: **Unsteady State Heat Transfer (Analytical Solutions)**

Professor Faith A. Morrison
Department of Chemical Engineering
Michigan Technological University

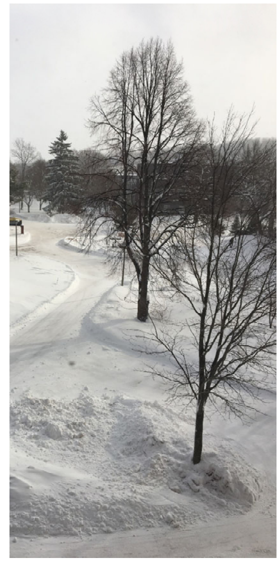
www.chem.mtu.edu/~fmorriso/cm3120/cm3120.html

We model the dynamics of unsteady state heat transfer because there are very practical problems that we can solve with such models.

Example:
When will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground?

1/30/19



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Unsteady State Heat Transfer: Dimensional Analysis

Engineering Modeling (complex systems)

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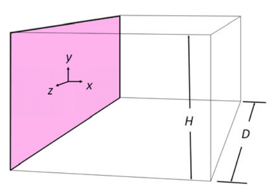
STEP ONE:

Idealized problem:
1D heat transfer in a semi-infinite solid



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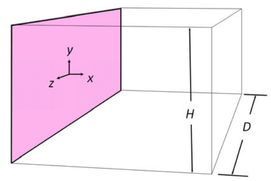
Unsteady State Heat Transfer: Dimensional Analysis

Develop a model:



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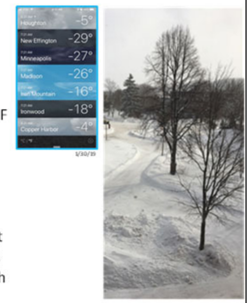
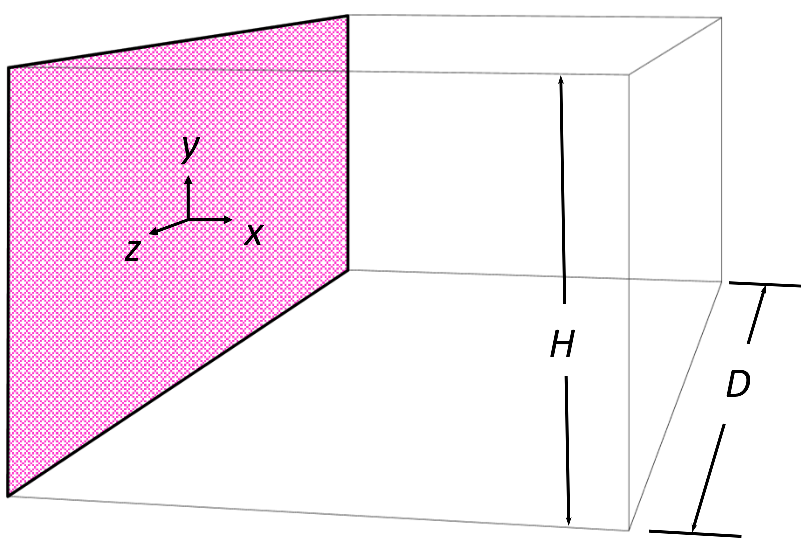


Example 1: **Unsteady Heat Conduction in a Semi-infinite solid**

A very long, very wide, very tall slab is initially at a temperature T_0. At time $t = 0$, the left face of the slab is exposed to a vigorously mixed gas at temperature T_1. What is the time-dependent temperature profile in the slab?

Unsteady State Heat Transfer

Example: Unsteady Heat Conduction in a Semi-infinite solid

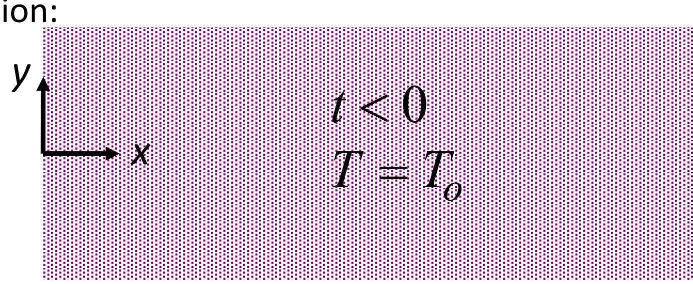


H, D, very large

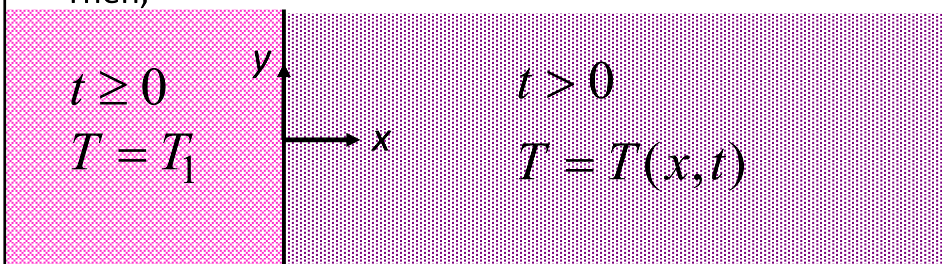
1D Heat Transfer: Unsteady State

Initial Condition:

$$t < 0$$
$$T = T_o$$



Then,

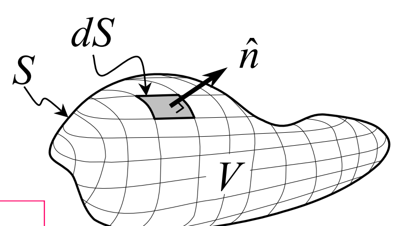


1D Heat Transfer: Unsteady State

General Energy Transport Equation

(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, *V*, enclosed by a surface, *S*.



Gibbs notation:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

see handout for component notation

1D Heat Transfer: Unsteady State

General Energy Transport Equation

(microscopic energy balance)

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t}+\underline{v}\cdot\nabla T\right)=k\nabla^2 T+S$$

rate of change

convection

conduction (all directions)

source (energy generated per unit volume per time)

velocity must satisfy equation of motion, equation of continuity

see handout for component notation

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Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t}+\underline{v}\cdot\nabla T\right)=\frac{k}{\rho\hat{C}_p}\nabla^2 T+\frac{S}{\rho\hat{C}_p}$$

thermal diffusivity $\alpha\equiv\frac{k}{\rho\hat{C}_p}$

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}=\frac{k}{\rho\hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{S}{\rho\hat{C}_p}$$

Cylindrical (rθz) coordinates:

$$\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}=\frac{k}{\rho\hat{C}_p}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{S}{\rho\hat{C}_p}$$

Spherical (rθϕ) coordinates:

$$\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}=\frac{k}{\rho\hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2}\right.$$

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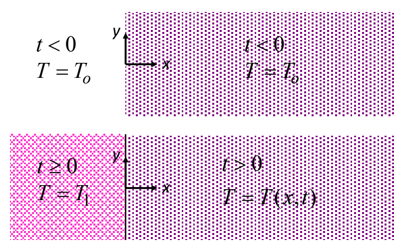
1D Heat Transfer: Unsteady State

Example 1: **Unsteady Heat Conduction in a Semi-infinite solid**

A very long, very wide, very tall slab is initially at a temperature T_0. At time $t = 0$, the left face of the slab is exposed to a vigorously mixed gas at temperature T_1. What is the time-dependent temperature profile in the slab?

Newton's law of cooling BC's:

$$|q_x| = hA\left|T_{bulk} - T_{surface}\right|$$



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1D Heat Transfer: Unsteady State

Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$$ thermal diffusivity

what are the boundary conditions? initial conditions?

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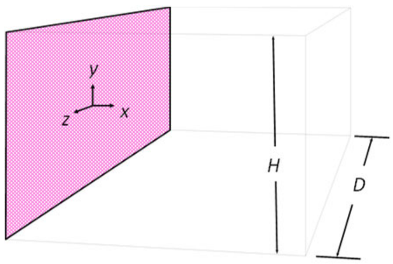
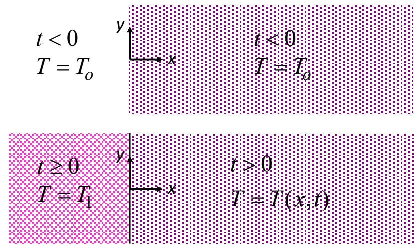
1D Heat Transfer: Unsteady State

Example: Unsteady Heat Conduction in a Semi-infinite solid



Initial Condition:



You try.

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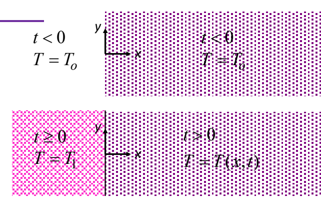
1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha \left(\frac{\partial^2 T}{\partial x^2}\right)$$

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

Initial condition: $t = 0 \quad T = T_0 \quad \forall \ x$

Boundary conditions:

$$x = 0 \quad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \quad t > 0$$

$$x = \infty \qquad T = T_0 \qquad \forall \ t$$

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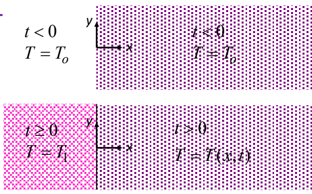
1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

thermal diffusivity

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Initial condition: $t = 0 \quad T = T_0 \quad \forall\ x$ "for all x"

Boundary conditions:

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"for all t"

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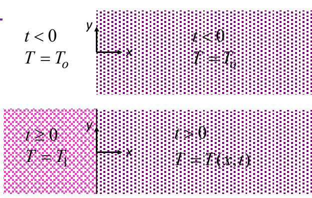
1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



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$$x = \infty \quad T = T_0 \quad \forall\ t$$

The solution of the PDE is obtained by combination of variables.

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

See text WRF p284

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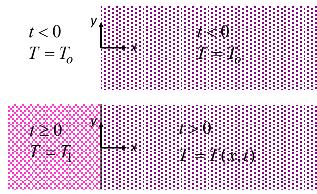
Unsteady State Heat Conduction in a Semi-Infinite Slab



Solution:

$$\frac{T-T_0}{T_1-T_0}=\operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

$$Y\equiv\frac{(T_1-T)}{(T_1-T_0)}$$

$$1-Y=\frac{(T-T_0)}{(T_1-T_0)}$$

$$\beta\equiv\frac{h\sqrt{\alpha\,t}}{k}\qquad \zeta\equiv\frac{x}{2\sqrt{\alpha\,t}}$$

complementary error function of y *(a standard function in Excel)*

$$\operatorname{erfc}(y)\equiv 1-\operatorname{erf}(y)$$

error function of y

$$\operatorname{erf}(y)\equiv\frac{2}{\sqrt{\pi}}\int_0^y e^{-(y')^2}\,dy'$$

- **Geankoplis 4th ed., eqn 5.3-7, page 363**
- **WRF, eqn 18-21, page 286**

thermal diffusivity $\alpha\equiv\frac{k}{\rho\hat{C}_p}$

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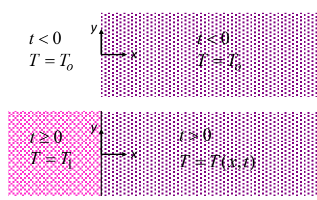
Unsteady State Heat Conduction in a Semi-Infinite Slab



Solution:

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$$\beta\equiv\frac{h\sqrt{\alpha\,t}}{k}\qquad \zeta\equiv\frac{x}{2\sqrt{\alpha\,t}}$$

To make this solution easier to use, we can plot it.

complementary error function of y

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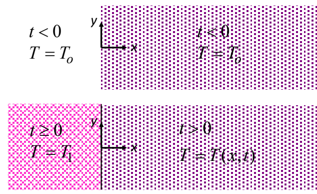
Unsteady State Heat Conduction in a Semi-Infinite Slab



This:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta + \beta)$$

Versus this: $\zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$

To make this solution easier to use, we can plot it.

At various values of this:

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

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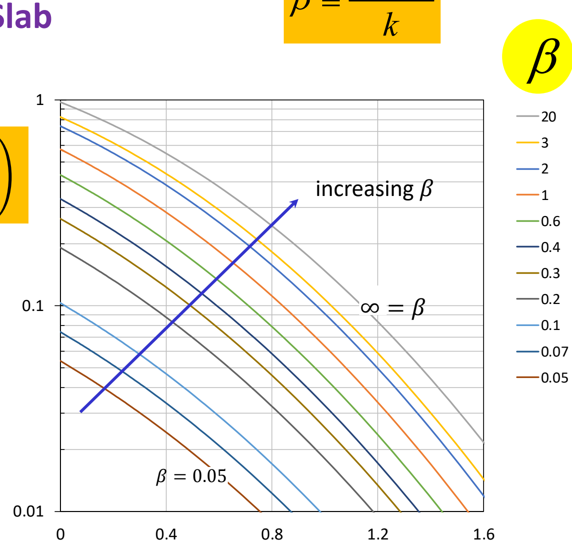
Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

$$1 - Y = \left(\frac{T - T_0}{T_1 - T_0}\right)$$



$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

Plot design after Geankoplis 4th ed., Figure 5.3-3, page 364

$$\zeta = \frac{x}{2\sqrt{\alpha\, t}}$$

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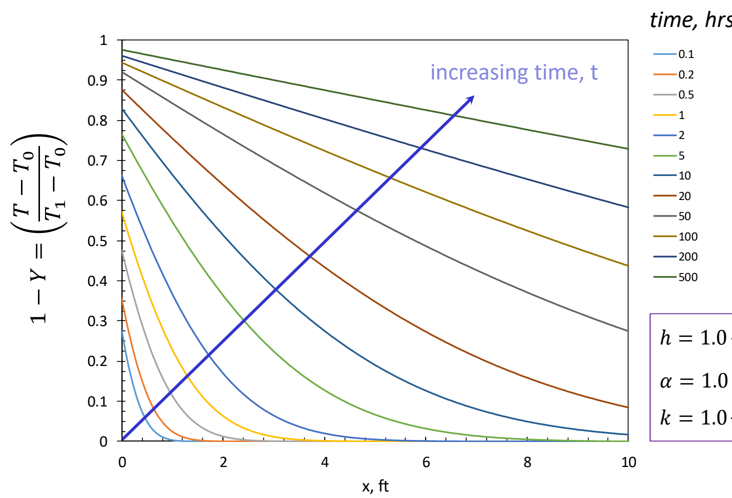
1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the solution directly (evaluated in Excel)

Unsteady State Heat Conduction in a Semi-Infinite Slab



$h = 1.0 \frac{btu}{h ft^2 F}$

$\alpha = 1.0 \frac{ft^2}{h}$

$k = 1.0 \frac{btu}{h} ft F$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

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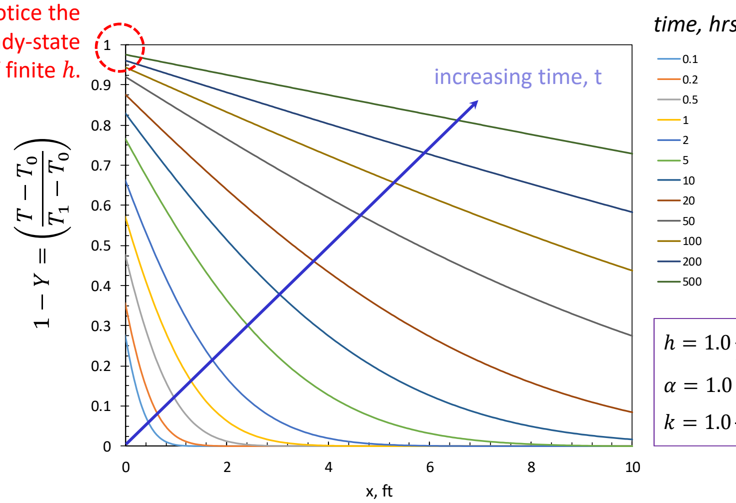
1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the solution directly (evaluated in Excel)

Unsteady State Heat Conduction in a Semi-Infinite Slab

Notice the steady-state effect of finite h.



$h = 1.0 \frac{btu}{h ft^2 F}$

$\alpha = 1.0 \frac{ft^2}{h}$

$k = 1.0 \frac{btu}{h} ft F$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

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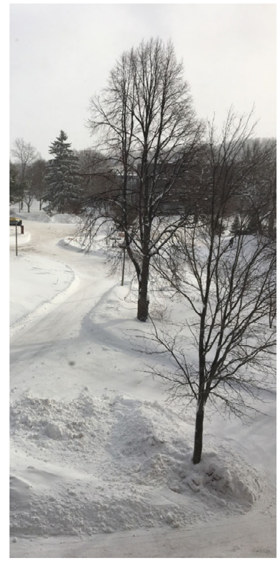
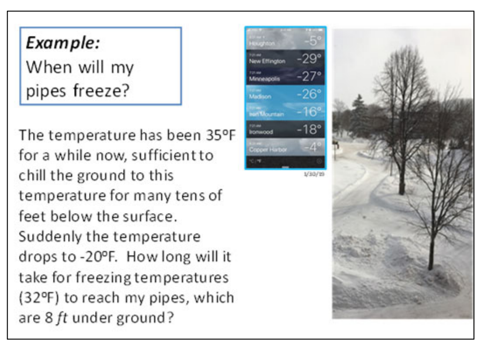
Example:
When will my pipes freeze?

1/30/19



The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab



We need the appropriate physical property data for the soil.

$$h = 2.0 \frac{BTU}{h\ ft^2\ {}^oF}$$

$$\alpha_{soil} = 0.018 \frac{ft^2}{h}$$

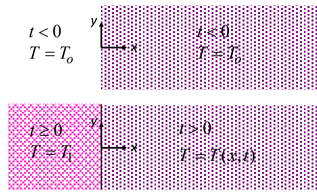
$$k_{soil} = 0.5 \frac{BTU}{h\ ft\ {}^oF}$$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Geankoplis 4th ed.

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{T-T_0}{T_1-T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

$$\zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

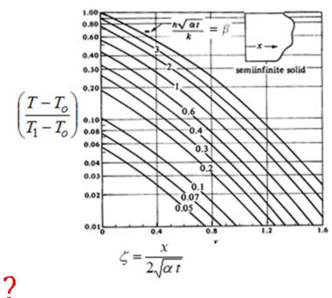
Both ζ and β depend on time

$T_0 = ?$

$T_1 = ?$

$T = ?$

$$\frac{T-T_0}{T_1-T_0} = ?$$



$$Y \equiv \frac{(T_1-T)}{(T_1-T_0)}$$

$$1-Y = \frac{(T-T_0)}{(T_1-T_0)}$$

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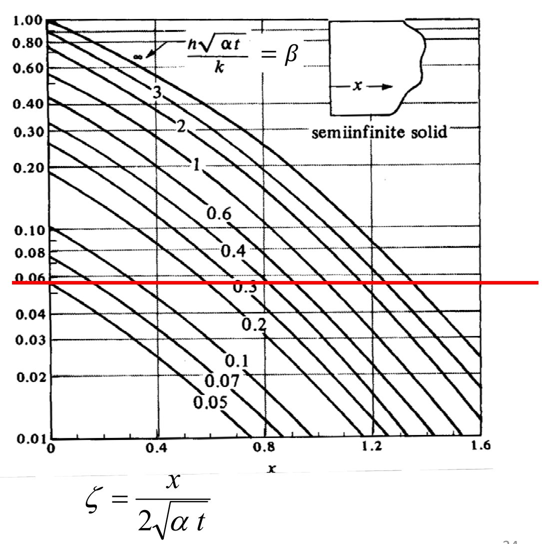
Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

$$1-Y = \left(\frac{T-T_0}{T_1-T_0}\right)$$

You try.



Geankoplis 4th ed., Figure 5.3-3, page 364

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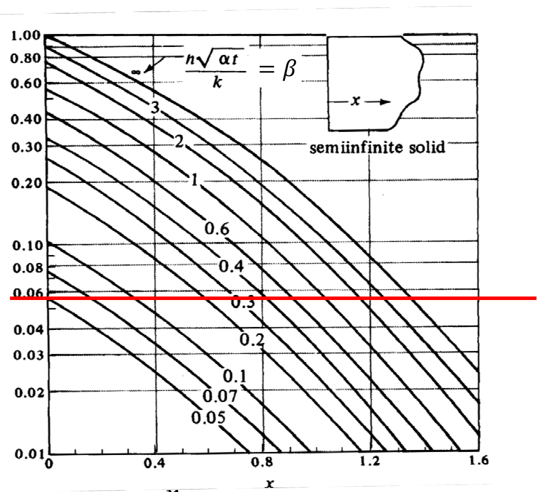
Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

$$1 - Y = \left(\frac{T - T_0}{T_1 - T_0}\right)$$



$$\zeta = \frac{x}{2\sqrt{\alpha\, t}}$$

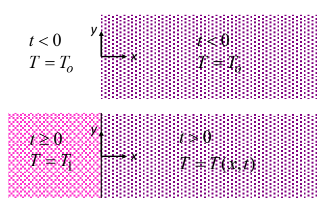
Solution:

Guess large β)
(Interative solution)

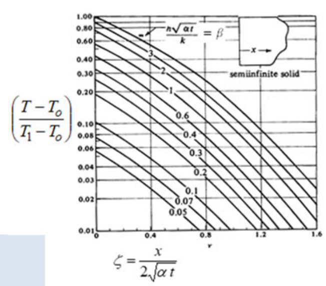
Geankoplis 4th ed., Figure 5.3-3, page 364

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta + \beta)$$



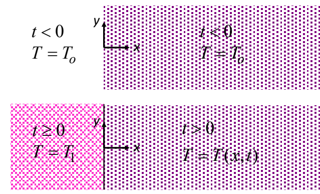
Answer:
$t \approx 480\ hours \approx 20\ days$

$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab



Or, use Excel. (How exactly?)

$$\frac{T-T_0}{T_1-T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

$$\zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

You try.

T0=		
T1=		
T=		
h=		
alpha=		
k=		
x=		

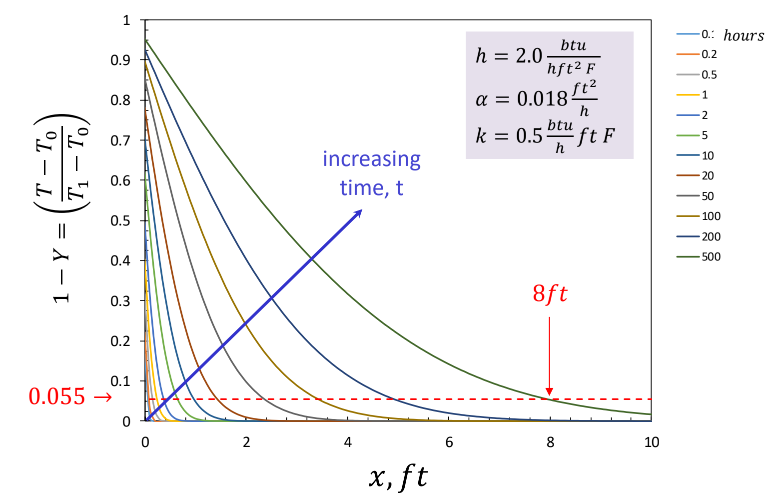
Answer:
$t = 21.2\ days$
$\beta = 12.1$

pages.mtu.edu/~fmorriso/cm310/2019PracticeProblemsInHeatTransfer%28Geankoplis%29.pdf

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the evolution of the model directly (evaluated in Excel)

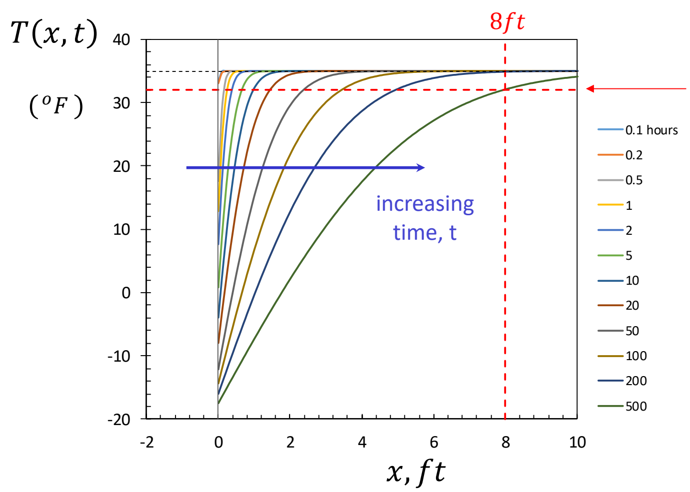


aa_solutions/Unsteady semi infinite solid plots.xlsx

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the evolution of the model directly (evaluated in Excel)



$h = 2.0 \frac{btu}{hft^2\,F}$

$\alpha = 0.018 \frac{ft^2}{h}$

$k = 0.5 \frac{btu}{h} ft\, F$

aa_solutions/Unsteady semi infinite solid plots.xlsx

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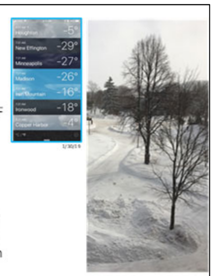
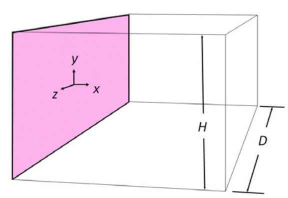
Solution Summary:

Example: When will my pipes freeze?

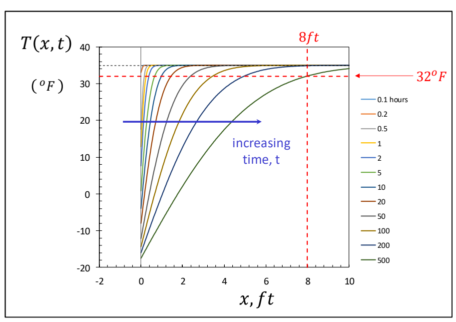
The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 ft under ground?





Answer:

$t = 509\ hours \approx 21\ days$



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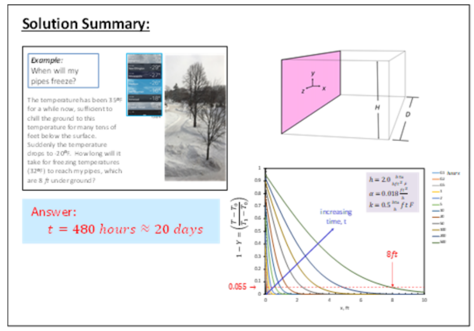
We used unsteady state heat transfer modeling to solve one practical problem.



What can we do to extend these methods to a wider class of problems?

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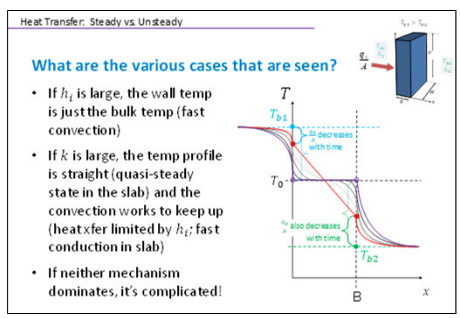
Back to this:

What is our usual strategy for complex phenomena?

Answer: Dimensional Analysis



Engineering Modeling (complex systems)

- ✓ Choose an idealized problem and solve it
- ✓ From insight obtained from **ideal** problem, identify governing equations of **real** problem
- Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
- Design experiments to test modeling thus far
- Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness lengthscale)
- Design additional experiments
- Iterate until useful correlations result

✓ *Let's nondimensionalize the governing equations and BCs.*

✓ *Let's sort out the various unsteady cases.*

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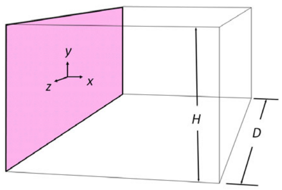
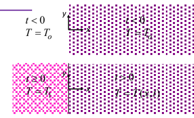
Let's nondimensionalize the governing equations and BCs.
Let's sort out the various cases.



1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t}=\frac{k}{\rho\hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right)=\alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

thermal diffusivity $\alpha\equiv\frac{k}{\rho\hat{C}_p}$

Initial condition: $t=0 \quad T=T_0 \quad \forall\ x$

Boundary conditions:

$x=0 \quad q_x=hA(T_1-T) \quad t>0$

$x=\infty \quad T_0 \quad \forall\ t$

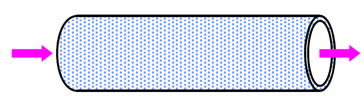
(Review:
***How did we do this before*?)**

CM3110 REVIEW

Method:

- Identify the governing equation(s)
- Choose "typical" values (*scale factors*)
- Use them to scale the equations

We'll modify our solution for
Convective Heat Transfer



Pipe flow

Dimensional Analysis

non-dimensional variables:

time:	position:	velocity:	driving force:
$t^*\equiv\frac{tV}{D}$	$r^*\equiv\frac{r}{D}$	$v_z^*\equiv\frac{v_z}{V}$	$P^*\equiv\frac{P}{\rho V^2}$
	$z^*\equiv\frac{z}{D}$	$v_r^*\equiv\frac{v_r}{V}$	$g_z^*\equiv\frac{g_z}{g}$
		$v_\theta^*\equiv\frac{v_\theta}{V}$	

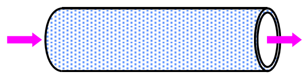
Energy

non-dimensional variables:

position:	temperature:	source:
$r^*\equiv\frac{r}{D}$	$T^*\equiv\frac{T-T_o}{(T_1-T_o)}$	$S^*\equiv\frac{S}{S_0}$
$z^*\equiv\frac{z}{D}$		

We'll modify our solution for

Convective Heat Transfer



Pipe flow

Dimensional Analysis

non-dimensional variables:

time: $t^* \equiv \frac{tV}{D}$

position: $r^* \equiv \frac{r}{D}$, $z^* \equiv \frac{z}{D}$

velocity: $v_z^* \equiv \frac{v_z}{V}$, $v_r^* \equiv \frac{v_r}{V}$, $v_\theta^* \equiv \frac{v_\theta}{V}$

driving force: $P^* \equiv \frac{P}{\rho V^2}$, $g_z^* \equiv \frac{g_z}{g}$

Energy

non-dimensional variables:

position: $r^* \equiv \frac{r}{D}$, $z^* \equiv \frac{z}{D}$

temperature: $T^* \equiv \frac{T - T_o}{(T_1 - T_o)}$

source: $S^* \equiv \frac{S}{S_0}$

Slight problem: We need to nondimensionalize t for the unsteady case also, but there is no characteristic velocity in thermal conduction in a solid.

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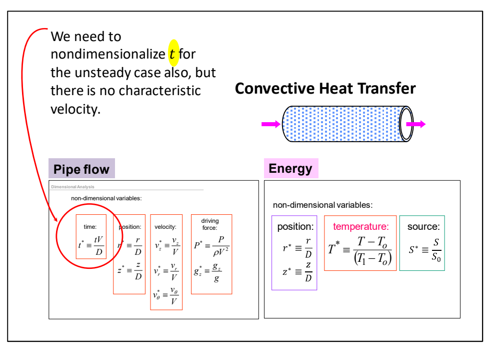
Choice:

For the unsteady case we'll choose a characteristic time based on the thermal diffusivity, α.



$$t^* \equiv \frac{\alpha t}{D^2}$$

This dimensionless time is called Fourier number Fo.

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

(Appears in the energy balance)

$$\frac{D^2}{\alpha} = \text{thermal diffusion time} = \frac{D^2 \rho \hat{C}_p}{k}$$

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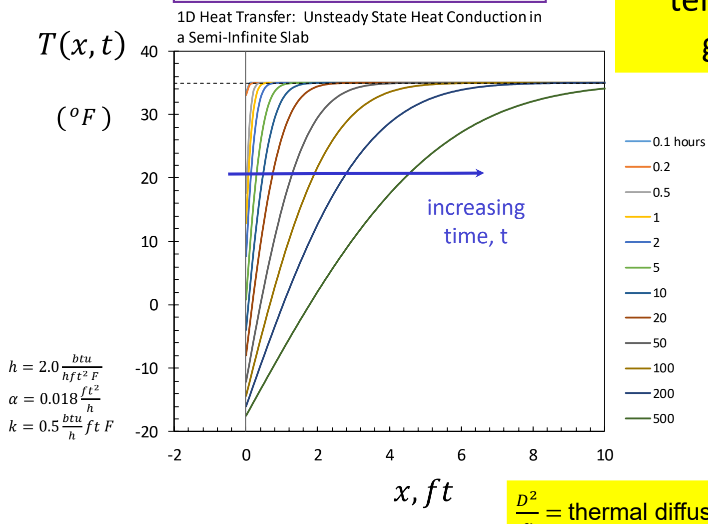
Energy is *diffusing* **down** the temperature gradient

Example: When will my pipes freeze?



thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

$\frac{D^2}{\alpha}$ = thermal diffusion time = $\frac{D^2 \rho \hat{C}_p}{k}$

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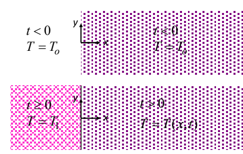
Dimensional Analysis, Unsteady State Convection

Non-dimensionalize (eqns, BCs)

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

$$q_x = -k \frac{\partial T}{\partial x} = hA(T_1 - T)$$



non-dimensional variables:

position: $x^* \equiv \frac{x}{D}$

temperature: $Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$

time: $t^* \equiv \frac{\alpha t}{D^2}$

This dimensionless time is called Fourier number Fo.

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

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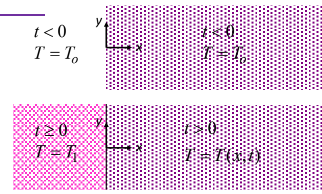
1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial Y}{\partial t^*} = \frac{\partial^2 Y}{\partial x^{*2}}$$

temperature:

$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

Initial condition: $t^* = 0 \quad Y = 1 \quad \forall\, x^*$

Boundary conditions: $x^* = \infty \quad Y = 1 \quad \forall\, t^*$

$$x^* = 0 \quad \frac{\partial Y}{\partial x^*} = \mathrm{Bi}\, Y \quad t^* > 0$$

$$\mathrm{Bi} \equiv \frac{hD}{k}$$

$$\mathrm{Bi} - \text{Biot Number} = \frac{hD}{k}$$

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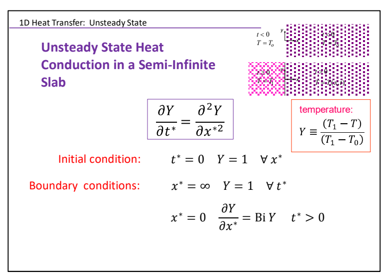
In dimensionless form, we see that this problem reduces to

$$Y = Y\left(\frac{x}{D}, \mathrm{Fo}, \mathrm{Bi}\right)$$



Dimensionless quantities:

$$Y = \frac{(T_1 - T)}{(T_1 - T_0)}$$ **Y** (dimensionless temperature interval)

$$t^* = \mathrm{Fo} = \frac{\alpha t}{D^2}$$ **Fourier number** (dimensionless time)

$$x^* = \frac{x}{D}$$

$$\mathrm{Bi} = \frac{hD}{k}$$

Biot number (pronounced BEE-OH)
Ratio of heat transfer resistance at the boundary to resistance in the solid. This is a transport issue.

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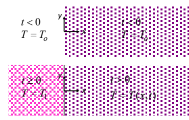
Because we can solve this problem analytically, **we can confirm** that the dimensional analysis is correct:

Solution:

Unsteady State Heat Conduction in a Semi-Infinite Slab



Solution:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

+

$$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$$

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

=

$$1 - Y = \operatorname{erfc}\left(\frac{x}{D}\frac{1}{2}\frac{1}{\sqrt{\text{Fo}}}\right) - e^{\text{Bi}\left(\frac{x}{D}\right)+\text{Bi}^2\text{Fo}}\operatorname{erfc}\left(\sqrt{\text{Fo}}\left(\text{Bi} + \frac{x}{D}\frac{1}{\text{Fo}}\right)\right)$$

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Unsteady State Heat Transfer in a Body

Two Additional Dimensionless Numbers

$$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

Dimensionless Numbers (momentum, energy, mass)

Re – Reynolds $= \frac{\rho V D}{\mu} = \frac{VD}{\nu}$ Fr – Froude $= \frac{V^2}{gD}$ Pe – Péclet$_h$ = RePr $= \frac{\hat{C}_p \rho V D}{k} = \frac{VD}{\alpha}$ Pe – Péclet$_m$ = ReSc $= \frac{VD}{D_{AB}}$	These numbers from the governing equations tell us about the relative importance of the terms they precede in the microscopic balances (***scenario*** *properties*).
Pr – Prandtl $= \frac{\hat{C}_p \mu}{k} = \frac{\nu}{\alpha}$ Sc – Schmidt = LePr $= \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$ Le – Lewis $= \frac{\alpha}{D_{AB}}$	These numbers compare the magnitudes of the diffusive transport coefficients ν, α, D_{AB} (***material*** *properties*).
f – Friction Factor $= \frac{\mathcal{F}_{drag}}{\left(\frac{1}{2}\rho V^2\right)A_c}$ Nu – Nusselt $= \frac{hD}{k}$ Sh – Sherwood $= \frac{k_m D}{D_{AB}}$	These numbers are defined to help us build transport data correlations based on the fewest number of grouped (dimensionless) variables (***scenario*** *properties*).

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

Scales the time evolution of the temperature profile relative to the material's thermal properties, $\alpha = k/\rho\hat{C}_p$.

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Dimensional Analysis in Unsteady State Heat Transfer

Warning!

Note Two Different Numbers

with completely different purposes and meanings
but confusingly similar definitions

$$\text{Bi} - \text{Biot Number} = \frac{hD}{k} = \frac{hD_{\text{body}}}{k_{\text{body}}}$$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h) for a body in contact with a moving fluid.

$$\text{Nu} - \text{Nusselt Number} = \frac{hD}{k} = \frac{hD_{flow}}{k_{fluid}}$$

Dimensionless heat transfer coefficient in convection. Quantifies the physics in the moving fluid and how this results in a resistance to heat transfer, captured in the heat transfer coefficient.

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$$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, <u>the surface temperature equals the bulk temperature</u>; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by <u>both</u> conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is <u>uniform in a finite body</u>; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

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$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat tran… the…

At m… affe… bod… the surface.

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

High Bi:

dominates

Low Bi: high k, low h

When the temperature is uniform in the body, we can do a macroscopic energy balance to solve many problems of interest. This is called a "lumped parameter analysis."

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$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by… body and t… the surface…

Moderate Bi:

At low Bi, t… uniform in… transfer is limited by rate of heat transfer to the surface (h).

high k, low h

When the wall temperature and the bulk temperature are equal, the microscopic energy balance is easier to carry out (temperature boundary conditions).

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Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

When both processes affect the outcomes, the full solution may be necessary. For uniform starting temperatures, the solutions are published.

the body. | high h

At moderate Bi, heat transfer is affected by <u>both</u> conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is <u>uniform in a finite body</u>; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

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NEXT: Talk about the three cases

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, <u>the surface temperature equals the bulk temperature</u>; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by <u>both</u> conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is <u>uniform in a finite body</u>; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

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