HIGH ACCURACY METHOD FOR TURBULENT FLOW PROBLEMS

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Abstract. We present a method of high order temporal and spatial accuracy for flow problems with high Reynolds number. The method presented is stable, computationally cheap and gives an accurate approximation to the quantities sought. The direct numerical simulation of turbulent flows is computationally expensive or not even feasible. Hence, the method employs turbulence modeling. The two key ingredients are the temporal deferred correction, combined with the family of Approximate Deconvolution models, which allows for arbitrarily high order of accuracy in both spatial and temporal variables. We prove stability and accuracy for the two-step method; the method is shown to be second order accurate in time and in the filtering width.

Key words. turbulence modeling, deferred correction, approximate deconvolution, high accuracy.

1. Introduction.

Direct numerical simulation of a $3d$ turbulent flow is often not computationally economical or even feasible. On the other hand, the largest structures in the flow (containing most of the flow's energy) are responsible for much of the mixing and most of the flow's momentum transport. This led to various numerical regularizations; one of these is Large Eddy Simulation (LES) [S01], [J04], [BIL06]. It is based on the idea that the flow can be represented by a collection of scales with different sizes, and instead of trying to approximate all of them down to the smallest one, one defines a filter width $\delta > 0$ and computes only the scales of size bigger than δ (large scales), while the effect of the small scales on the large scales is modeled. This reduces the number of degrees of freedom in a simulation and represents accurately the large structures in the flow.

Many different LES regularizations have been proposed and studied; we consider the family of Approximate Deconvolution Models (ADMs), which allow for arbitrarily high spatial accuracy. These models were introduced by Stolz and Adams in [AS] and extensively studied - see, e.g., [SAK, MM07, LL06, LaTr07, LaTr08, DE06]. In addition to having other advantages, the ADMs were applied in different areas, including the magnetohydrodynamics and compressible Navier-Stokes equations. The high spatial accuracy was achieved, but the time discretization was always performed by a low order backward Euler method or a Crank-Nicolson method, which introduces non-physical oscillations. But since solving the Navier-Stokes equations is computationally expensive even with turbulence models, one usually cannot choose the time step significantly smaller than the mesh size. Hence, one of the main advantages of the ADMs - the increased spatial accuracy - cannot be taken full advantage of, unless it is combined with a high accuracy time discretization. The proposed method also needs to be unconditionally stable and allow for explicit-implicit implementations with different time scales.

To that end, we employ the spectral deferred correction (SDC) method, proposed for stiff ODEs by Dutt et al., [DGR00] and further developed by Minion et al., (see [M03, M04, BLM03] and the references therein). The SDC methods were studied and compared to intrinsically high-order methods such as additive Runge-Kutta methods

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and linear multistep methods based on BDFs, with conclusion that the SDC methods are at least comparable to the latter. In addition, achieving high accuracy for the turbulent NSE using Runge-Kutta-based methods is very expensive, and the BDF-based methods typically don't perform well in problems where relevant time scales associated with different terms in the equation are widely different (see, e.g., [BLM03] for an example of advection-diffusion-reaction problem, where the SDC is the best choice of high accuracy temporal discretization).

In a "classical" understanding, the deferred correction approach to solving ODEs is based on replacing the original ODE with the corresponding Picard integral equation, discretizing the time interval, solving the integral equation approximately and then correcting the solution by solving a sequence of error equations on the same grid with the same scheme (see [DGR00] and [M03] for the detailed mathematical presentation of SDC). In the case of turbulence modeling, however, we face a new problem: the traditional Method Of Lines doesn't lead to the equation of the form

$$\phi_t = F(t, \phi(t))$$

as in the typical SDC setting. In the case of turbulence modeling with the approximate deconvolution models employed, both the energy and the energy dissipation of the flow are modified, therefore yielding

$$\phi_t + A\phi_t = F(t, \phi(t)).$$

We perform full numerical analysis of the method, proving both theoretically and computationally that the increased accuracy of classical SDC methods is achieved in the case of turbulent flow modeling as well. Therefore, we obtain an efficient method that gives a stable and high accuracy approximation to a solution of turbulent Navier-Stokes equations. The efficiency of the method is obtained by utilizing a turbulent model with high spatial accuracy and significantly less number of degrees of freedom than in the case of direct numerical simulations; and we also achieve high temporal accuracy by applying a stable and computationally cheap Backward Euler method.

We begin by introducing the simplest approximate deconvolution model of turbulence (see, e.g., [MM07]). For that, a filtering operator needs to be chosen, which commutes with differentiation under the periodic boundary conditions[4]. Throughout this paper, we shall use the selfadjoint filtering operator $A = I - \delta^2\Delta$ defined in Section 2.

The model, written in the traditional variational formulation, seeks $(w, p) \in ((X \bigcap H^2(\Omega)), Q)$ such that for any $(v, q) \in ((X \bigcap H^2(\Omega)), Q)$

$$\begin{aligned} (Aw_t, v) + \nu(A\nabla w, \nabla v) + b^*(w, w; v) - (p, \nabla \cdot v) &= (f, v), \\ (\nabla \cdot w, q) &= 0, \end{aligned} \tag{1.1}$$

where w approximates the averaged velocity $\bar{u}$. Note, however, that with the given choice of the filtering operator A we get a fourth order term $\nu\delta^2(\Delta w, \Delta v)$ in (1.1). In order to avoid using C^1 elements, we follow [MM07] and employ the mixed variational formulation: find $(w^h, \zeta^h, q^h) \in (X^h, X^h, Q^h)$ such that for any $(v^h, \xi^h, \chi^h) \in (X^h, X^h, Q^h)$

[4] In order to keep the analysis from becoming too technical, we assume periodic boundary conditions; however, in the computational section we consider the wave propagation model and numerically demonstrate that the theoretical results hold even in the case of Dirichlet boundary conditions.

$$
\begin{aligned}
(w_t^h, v^h) + \delta^2(\nabla w_t^h, \nabla v^h) + \nu(\nabla w^h, \nabla v^h) + \nu\delta^2(\nabla\zeta^h, \nabla v^h) \\
+ b^*(w^h, w^h; v^h) - (q^h, \nabla \cdot v^h) = (f, v^h), \\
(\nabla w^h, \nabla \xi^h) = (\zeta^h, \xi^h), \\
(\nabla \cdot w^h, \chi^h) = 0.
\end{aligned} \tag{1.2}
$$

Based on this mixed formulation ADM, we now proceed to formulating the high order accurate method, utilizing the deferred correction approach. The two-step deferred correction method computes (w_1^h, q_1^h) and (w_2^h, q_2^h) - two consecutive approximations for the averaged velocity and averaged pressure $(\bar{u}, \bar{p})$. These approximations satisfy the following equations for $(w_1^{h,n+1}, \zeta_1^{h,n+1}, q_1^{h,n+1})$, $(w_2^{h,n+1}, \zeta_2^{h,n+1}, q_2^{h,n+1}) \in (X^h, X^h, Q^h), \forall (v^h, \xi^h, \chi^h) \in (X^h, X^h, Q^h)$ at $t = t_{n+1}$, $n \geq 0$, with $k := \Delta t = t_{i+1} - t_i$:

$$
\begin{aligned}
&(\frac{w_1^{h,n+1} - w_1^{h,n}}{k}, v^h) + \delta^2(\nabla(\frac{w_1^{h,n+1} - w_1^{h,n}}{k}), \nabla v^h) + \nu(\nabla w_1^{h,n+1}, \nabla v^h) \qquad (1.3a)\\
&+ \nu\delta^2(\nabla\zeta_1^{h,n+1}, \nabla v^h) + b^*(w_1^{h,n+1}, w_1^{h,n+1}, v^h) - (q_1^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h),\\
&(\frac{w_2^{h,n+1} - w_2^{h,n}}{k}, v^h) + \delta^2(\nabla(\frac{w_2^{h,n+1} - w_2^{h,n}}{k}), \nabla v^h) + \nu(\nabla w_2^{h,n+1}, \nabla v^h) \qquad (1.3b)\\
&+ \nu\delta^2(\nabla\zeta_2^{h,n+1}, \nabla v^h) + b^*(w_2^{h,n+1}, w_2^{h,n+1}, v^h) - (q_2^{h,n+1}, \nabla \cdot v^h)\\
&= (\frac{f(t_{n+1}) + f(t_n)}{2}, v^h) + \frac{\nu}{2}k(\nabla(\frac{w_1^{h,n+1} - w_1^{h,n}}{k}), \nabla v^h)\\
&+ \frac{\nu}{2}\delta^2 k(\nabla(\frac{\zeta_1^{h,n+1} - \zeta_1^{h,n}}{k}), \nabla v^h) + \frac{1}{2}(b^*(w_1^{h,n+1}, w_1^{h,n+1}, v^h) - b^*(w_1^{h,n}, w_1^{h,n}, v^h)),\\
&(\nabla w_j^{h,n+1}, \nabla \xi^h) = (\zeta_j^{h,n+1}, \xi^h),\ j = 1, 2, \qquad (1.3c)\\
&(\nabla \cdot w_j^{h,n+1}, \chi^h) = 0,\ j = 1, 2. \qquad (1.3d)
\end{aligned}
$$

where $b^*(\cdot, \cdot, \cdot)$ is the explicitely skew-symmetrized trilinear form, defined below. Note that the second step utilizes the same Backward Euler time discretization as in the first step; only the right hand side is modified by a known quantity (a known solution from the first step). This results in the computational attractiveness of the method - computing two low order accurate approximations is much less costly (especially for very stiff problems) than computing a higher order approximation once.

The initial value approximations are taken to be $w_1^{h,0} = w_2^{h,0} = \bar{u}_0^s$, where u_0^s is the modified Stokes projection of u_0 onto the space V^h of discretely divergence-free functions (this projection and this space are defined in section 2).

The paper is organized as follows. In Section 2 we introduce the necessary notations and preliminary results. In Sections 3 and 4 we consider the first and second approximations (respectively) to the averaged true solution. In Section 3 we prove unconditional stability and accuracy of the first (backward Euler) approximation; we then use these results in Section 4 to verify stability and increased accuracy of the second (correction step) approximation.

2. Mathematical preliminaries and notations. Throughout this paper the norm $\|\cdot\|$ will denote the usual $L^2(\Omega)$-norm of scalars, vectors and tensors, induced by the usual L^2 inner-product, denoted by $(\cdot, \cdot)$. The space that velocity (at time t)

belongs to, is

$$X = H_0^1(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d\times d} \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

with the norm $\|v\|_X = \|\nabla v\|$. The space dual to X, is equipped with the norm

$$\|f\|_{-1} = \sup_{v\in X} \frac{(f, v)}{\|\nabla v\|}.$$

The pressure (at time t) is sought in the space

$$Q = L_0^2(\Omega) = \{q : q \in L^2(\Omega), \int_\Omega q(x)dx = 0\}.$$

Also introduce the space of weakly divergence-free functions

$$X \supset V = \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

For measurable $v : [0, T] \to X$, we define

$$\|v\|_{L^p(0,T;X)} = (\int_0^T \|v(t)\|_X^p dt)^{\frac{1}{p}},\ 1 \le p < \infty,$$

and

$$\|v\|_{L^\infty(0,T;X)} = ess \sup_{0\le t\le T} \|v(t)\|_X.$$

Define the trilinear form on $X \times X \times X$

$$b(u, v, w) = \int_\Omega u \cdot \nabla v \cdot w dx.$$

The following lemma is also necessary for the analysis

LEMMA 2.1. *There exist finite constants $M = M(d)$ and $N = N(d)$ s.t. $M \ge N$ and*

$$M = \sup_{u,v,w\in X} \frac{b(u, v, w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|} < \infty\ ,\ N = \sup_{u,v,w\in V} \frac{b(u, v, w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|} < \infty.$$

The proof can be found, for example, in [GR79]. The corresponding constants M^h and N^h are defined by replacing X by the finite element space $X^h \subset X$ and V by $V^h \subset X$, which will be defined below. Note that $M \ge \max(M^h, N, N^h)$ and that as $h \to 0$, $N^h \to N$ and $M^h \to M$ (see [GR79]).

Throughout the paper, we shall assume that the velocity-pressure finite element spaces $X^h \subset X$ and $Q^h \subset Q$ are conforming, have typical approximation properties of finite element spaces commonly in use, and satisfy the discrete inf-sup, or LBB^h, condition

$$\inf_{q^h\in Q^h} \sup_{v^h\in X^h} \frac{(q^h, \nabla \cdot v^h)}{\|\nabla v^h\|\|q^h\|} \ge \beta^h > 0, \tag{2.1}$$

where β^h is bounded away from zero uniformly in h. Examples of such spaces can be found in [GR79]. We shall consider $X^h \subset X$, $Q^h \subset Q$ to be spaces of continuous piecewise polynomials of degree r and $r - 1$, respectively, with $r \ge 1$.

The space of discretely divergence-free functions is defined as follows

$$V^h = \{v^h \in X^h : (q^h, \nabla \cdot v^h) = 0, \forall q^h \in Q^h\}.$$

The idea of approximate deconvolution modeling is based on the definition and properties of the following operator.

DEFINITION 2.2 (Approximate Deconvolution Operator). *For a fixed finite N, define the Nth approximate deconvolution operator G_N by*

$$G_N \phi = \sum_{n=0}^{N} (I - A_\delta^{-1})^n \phi,$$

where the averaging operator A_δ^{-1} is the differential filter: given $\phi \in L_0^2(\Omega)$, $\overline{\phi}^\delta \in H^2(\Omega) \cap L_0^2(\Omega)$ is the unique solution of

$$A_\delta \overline{\phi}^\delta := -\delta^2 \Delta \overline{\phi}^\delta + \overline{\phi}^\delta = \phi \quad \text{in } \Omega, \tag{2.2}$$

subject to periodic boundary conditions. Under periodic boundary conditions, this averaging operator commutes with differentiation.

LEMMA 2.3. *The operator G_N^i is compact, positive, and is an asymptotic inverse to the filter A_δ^{-1}, i.e., for very smooth ϕ and as $\delta \to 0$ satisfies*

$$\phi = G_N \overline{\phi}^\delta + (-1)^{N+1} \delta^{2N+2} \Delta^{N+1} A_\delta^{-(N+1)} \phi. \tag{2.3}$$

The proof of Lemma 2.3 can be found in [DE06].

We also define the following norm, induced by the deconvolution operator A:

$$\|\phi\|_A^2 = \|\phi\|^2 + \delta^2 \|\nabla \phi\|^2.$$

In the analysis we use the properties of the following Modified Stokes Projection (see [MM07]).

DEFINITION 2.4 (Modified Stokes Projection). *Define the Stokes projection operator P_S: $(X, X, Q) \to (X^h, X^h, Q^h)$, $P_S(u, \zeta, p) = (\tilde{u}, \tilde{\zeta}, \tilde{p})$, satisfying*

$$\begin{aligned} \nu(\nabla(u - \tilde{u}), \nabla v^h) + \nu\delta^2(\nabla(\zeta - \tilde{\zeta}), \nabla v^h) + (p - \tilde{p}, \nabla \cdot v^h) &= 0, \\ (\nabla(u - \tilde{u}), \nabla \xi^h) &= (\zeta - \tilde{\zeta}, \xi^h), \\ (\nabla \cdot (u - \tilde{u}), q^h) &= 0, \end{aligned} \tag{2.4}$$

for any $v^h \in X^h, \xi^h \in X^h, q^h \in Q^h$.

In (V^h, X^h, Q^h) this formulation reads: given $(u, \zeta, p) \in (X, X, Q)$, find $(\tilde{u}, \tilde{\zeta}) \in (V^h, X^h)$ satisfying

$$\begin{aligned} \nu(\nabla(u - \tilde{u}), \nabla v^h) + \nu\delta^2(\nabla(\zeta - \tilde{\zeta}), \nabla v^h) + (p - q^h, \nabla \cdot v^h) &= 0, \\ (\nabla(u - \tilde{u}), \nabla \xi^h) &= (\zeta - \tilde{\zeta}, \xi^h) \end{aligned} \tag{2.5}$$

for any $v^h \in V^h, \xi^h \in X^h, q^h \in Q^h$.

We give without proof the stability and accuracy results for the Modified Stokes projection (2.5). The proof can be found in [MM07]. Note also that the usual choice for the filtering width is $\delta = h$.

PROPOSITION 2.5 (Stability of the Stokes projection). *Let $\tilde{u}$, $\tilde{\zeta}$ satisfy (2.5) for the given u, ζ. The following bound holds*

$$\nu\|\nabla\tilde{u}\|^2 + \nu\delta^2\|\tilde{\zeta}\|^2 \leq C(\nu(1+\delta^2h^{-2})\|\nabla u\|^2 + \nu\delta^2\|\zeta\|^2 \tag{2.6}$$

$$+\nu^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2). \tag{2.7}$$

In the error analysis we shall use the error estimate of the Stokes projection (2.5).

PROPOSITION 2.6 (Error estimate for Stokes Projection). *Suppose the discrete inf-sup condition (2.1) holds. Then the error in the Stokes Projection satisfies*

$$\nu\|\nabla(u-\tilde{u})\|^2 + \nu\delta^2\|\zeta-\tilde{\zeta}\|^2 \tag{2.8}$$

$$\leq C[\nu(1+\delta^2h^{-2})\inf_{v^h\in V^h}\|\nabla(u-v^h)\|^2 + \nu\delta^2(1+\delta^2h^{-2})\inf_{\xi^h\in X^h}\|\zeta-\xi^h\|^2$$

$$+\nu^{-1}\inf_{q^h\in Q^h}\|p-q^h\|^2],$$

where C is a constant independent of h and Re.

Define the explicitly skew-symmetrized trilinear form

$$b^*(u,v,w) := \frac{1}{2}(u\cdot\nabla v, w) - \frac{1}{2}(u\cdot\nabla w, v).$$

The following estimate is easy to prove (see, e.g., [GR79]): there exists a constant $C = C(\Omega)$ such that

$$|b^*(u,v,w)| \leq C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|. \tag{2.9}$$

The proofs will require the sharper bound on the nonlinearity. This upper bound is improvable in $\mathbb{R}^2$.

LEMMA 2.7 (The sharper bound on the nonlinear term). *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. For all $u, v, w \in X$*

$$|b^*(u,v,w)| \leq C(\Omega)\sqrt{\|u\|\|\nabla u\|}\|\nabla v\|\|\nabla w\|.$$

Proof. See [GR79]. ◻

We will also need the following inequalities: for any $u \in V$

$$\inf_{v\in V^h}\|\nabla(u-v)\| \leq C(\Omega)\inf_{v\in X^h}\|\nabla(u-v)\|, \tag{2.10}$$

$$\inf_{v\in V^h}\|u-v\| \leq C(\Omega)\inf_{v\in X^h}\|\nabla(u-v)\|. \tag{2.11}$$

The proof of (2.10) can be found, e.g., in [GR79], and (2.11) follows from the Poincare-Friedrich's inequality and (2.10).

Define also the number of time steps $N := \frac{T}{k}$.

We conclude the preliminaries by formulating the discrete Gronwall's lemma, see, e.g. [HR90]

LEMMA 2.8. *Let k, B, and $a_\mu, b_\mu, c_\mu, \gamma_\mu$, for integers $\mu \geq 0$, be nonnegative numbers such that:*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq k\sum_{\mu=0}^{n} \gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B \text{ for } n \geq 0.$$

Suppose that $k\gamma_\mu < 1$ for all μ, and set $\sigma_\mu = (1-k\gamma_\mu)^{-1}$. Then

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq e^{k\sum_{\mu=0}^{n}\sigma_\mu\gamma_\mu} \cdot [k\sum_{\mu=0}^{n} c_\mu + B].$$

3. Stability and accuracy of the first approximate solution. In this section we investigate the first approximate solution w_1^h, satisfying $(1.3a)$. We prove the unconditional stability of the solution and we perform the error analysis. We also prove the error estimate for the time difference of the error $\|\frac{\bar{u}(t_i)-w_1^{h,i}}{k}\|$, which is needed for the analysis of the correction step.

3.1. Stability of the Velocity. We start by proving the unconditional stability of the first approximation w_1^h, satisfying $(1.3a)$.

THEOREM 3.1 (Stability of the first approximation). *Let w_1^h satisfy (1.3a). Let $f \in L^2(0,T;H^{-1}(\Omega))$. Then for $n = 0, ..., N-1$*

$$\|w_1^{h,n+1}\|_A^2 + \nu k\Sigma_{i=0}^n\|\nabla w_1^{h,i+1}\|^2 + \nu\delta^2 k\Sigma_{i=0}^n\|\zeta_1^{h,i+1}\|^2 \leq \|w_1^{h,0}\|_A^2 + \frac{1}{\nu}k\Sigma_{i=0}^n\|f(t_{i+1})\|_{-1}^2.$$

Proof. Let $v^h = w_1^{h,n+1} \in V^h$ in $(1.3a)$; let also $\xi^h = \zeta_1^{h,n+1}, j = 1$ in $(1.3c)$. Since $b^*(u,v,v) = 0$, we obtain

$$\frac{\|w_1^{h,n+1}\|^2 - (w_1^{h,n}, w_1^{h,n+1})}{k} + \delta^2\frac{\|\nabla w_1^{h,n+1}\|^2 - (\nabla w_1^{h,n}, \nabla w_1^{h,n+1})}{k} \tag{3.1}$$
$$+\nu\|\nabla w_1^{h,n+1}\|^2 + \nu\delta^2\|\zeta_1^{h,n+1}\|^2 - (q_1^{h,n+1}, \nabla\cdot w_1^{h,n+1}) = (f(t_{n+1}), w_1^{h,n+1}).$$

Since $q_1^{h,n+1} \in Q^h$ and $w_1^{h,n+1} \in V^h$, the pressure term in (3.1) vanishes. Apply Cauchy-Schwartz and Young's inequalities; or the term in the right hand side use the definition of the dual norm.

$$\frac{\|w_1^{h,n+1}\|^2 - \|w_1^{h,n}\|^2}{2k} + \delta^2\frac{\|\nabla w_1^{h,n+1}\|^2 - \|\nabla w_1^{h,n}\|^2}{2k} \tag{3.2}$$
$$+\frac{\nu}{2}\|\nabla w_1^{h,n+1}\|^2 + \nu\delta^2\|\zeta_1^{h,n+1}\|^2 \leq \frac{1}{2\nu}\|f(t_{n+1})\|_{-1}^2.$$

Summing (3.2) over all time levels and multiplying by $2k$ completes the proof. ☐

3.2. Error estimates. In this section we explore the error estimates in approximating the averaged NSE velocity $\bar{u}$ by the solution w_1^h of $(1.3a)$. We also derive an estimate for the time difference of the error $\frac{e_1^{i+1}-e_1^i}{k}$. This result will be used in the following section.

Before we proceed to proving an estimate of the error $\bar{u} - w_1$, let's look at the variational formulation of the averaged momentum equation, satisfied by the averaged velocity $\bar{u}$.

$$(\bar{u}_t, v) + \nu(\nabla\bar{u}, \nabla v) + \overline{b^*(\bar{u}, \bar{u}; v)} - (\bar{p}, \nabla\cdot v) \tag{3.3}$$
$$= (\bar{f}, v) + \overline{b^*(\bar{u}, \bar{u}; v)} - \overline{b^*(u, u; v)}.$$

Applying the exact deconvolution operator $A = (I - \delta^2\Delta)$ to (3.3) we obtain

$$\begin{aligned}(A(\frac{\bar{u}(t_{n+1}) - \bar{u}(t_n)}{k}), v) + \nu(\nabla\bar{u}(t_{n+1}), \nabla v) + \nu\delta^2(\nabla\zeta(t_{n+1}), \nabla v)\\ +b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); v) - (p(t_{n+1}), \nabla\cdot v) = (f(t_{n+1}), v)\\ +(b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); v) - b^*(u(t_{n+1}), u(t_{n+1}); v))\\ +(A(\frac{\bar{u}(t_{n+1}) - \bar{u}(t_n)}{k} - \bar{u}_t(t_{n+1})), v),\\ (\nabla\bar{u}(t_{n+1}), \nabla\xi^h) = (\zeta(t_{n+1}), \xi^h),\ \forall\xi^h \in X^h.\end{aligned} \tag{3.4}$$

We are now ready to prove the accuracy of the first approximation.

THEOREM 3.2. *Let the time step satisfy* $k < \frac{\nu^3}{\max_{i=0,1,..,N}\|\nabla\bar{u}(t_i)\|^4}$. *Let also* $\bar{u} \in L_2(0,T; H_3(\Omega))$ *and* $\bar{u}_{tt} \in L_2(0,T; H_1(\Omega))$. *Then the error in the first approximation satisfies*

$$\begin{aligned}\|\bar{u}(t_{n+1}) - w_1^{h,n+1}\|_A^2 + \nu k\sum_{i=0}^{n}\|\nabla(\bar{u}(t_{i+1}) - w_1^{h,i+1})\|^2\\ +\nu\delta^2 k\sum_{i=0}^{n}\|\zeta(t_{i+1}) - \zeta_1^{h,i+1}\|^2 \le C(\nu,\bar{u})(k^2 + \delta^4\\ +k\sum_{i=0}^{n}((1+\delta^2 h^{-2})\inf_{v\in V^h}\|\nabla(\bar{u}(t_i) - v^i)\|^2 + \inf_{q\in Q^h}\|p(t_i) - q^i\|^2)).\end{aligned} \tag{3.5}$$

Proof. Subtract (1.3*a*) from (3.4). Decompose the error $e_1^i = \bar{u}(t_i) - w_1^{h,i} = \bar{u}(t_i) - \tilde{w}^i + \tilde{w}^i - w_1^{h,i} = \eta^i + \phi^i$, where $\tilde{w}^i$ is some approximation of $\bar{u}(t_i)$ in V_h. Then:

$$\begin{aligned}(A(\frac{\phi^{n+1} - \phi^n}{k}), v) + \nu(\nabla\phi^{n+1}, \nabla v)\\ +\nu\delta^2(\nabla(\zeta(t_{n+1}) - \zeta_1^{h,n+1}), \nabla v) = -(A(\frac{\eta^{n+1} - \eta^n}{k}), v)\\ -\nu(\nabla\eta^{n+1}, \nabla v) - b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); v) + b^*(w_1^{h,n+1}, w_1^{h,n+1}, v)\\ +(b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); v) - b^*(u(t_{n+1}), u(t_{n+1}); v))\\ -(p(t_{n+1}) - q_1^{h,n+1}, \nabla\cdot v) + (A(\frac{\bar{u}(t_{n+1}) - \bar{u}(t_n)}{k} - \bar{u}_t(t_{n+1})), v),\\ (\nabla\phi^{n+1}, \nabla\xi^h) + (\nabla\eta^{n+1}, \nabla\xi^h) = (\zeta(t_{n+1}) - \zeta_1^{h,n+1}, \xi^h).\end{aligned} \tag{3.6}$$

Choose $\tilde{w}^{n+1}$ to be the Modified Stokes Projection of $\bar{u}(t_{n+1})$, with $\tilde{\zeta}^{n+1}$ being the Modified Stokes Projection of $\zeta(t_{n+1})$. Choose $v = \phi^{n+1} \in V_h$ and $\xi^h = \tilde{\zeta}^{n+1} - \zeta_1^{h,n+1}$ in (3.6). Apply the Young's inequality to the first term in the left hand side; using the definition of Modified Stokes Projection (2.5) gives

$$\begin{aligned}
&\frac{\|\phi^{n+1}\|_A^2 - \|\phi^n\|_A^2}{2k} + \nu\|\nabla\phi^{n+1}\|^2 \qquad (3.7)\\
&+\nu\delta^2(\nabla(\tilde{\zeta}^{n+1} - \zeta_1^{h,n+1}), \nabla\phi^{n+1}) \le -(\frac{\eta^{n+1} - \eta^n}{k}, \phi^{n+1})\\
&-\delta^2(\nabla\left(\frac{\eta^{n+1} - \eta^n}{k}\right), \nabla\phi^{n+1}) - \nu\delta^2(\Delta\eta^{n+1}, \Delta\phi^{n+1})\\
&-b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1}) + b^*(w_1^{h,n+1}, \bar{u}(t_{n+1}); \phi^{n+1})\\
&-b^*(w_1^{h,n+1}, \bar{u}(t_{n+1}); \phi^{n+1}) + b^*(w_1^{h,n+1}, w_1^{h,n+1}; \phi^{n+1})\\
&+(b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1}) - b^*(u(t_{n+1}), u(t_{n+1}); \phi^{n+1}))\\
&+(\frac{\bar{u}(t_{n+1}) - \bar{u}(t_n)}{k} - \bar{u}_t(t_{n+1}), \phi^{n+1})\\
&+\delta^2(\nabla\left(\frac{\bar{u}(t_{n+1}) - \bar{u}(t_n)}{k} - \bar{u}_t(t_{n+1})\right), \nabla\phi^{n+1}),\\
&(\nabla\phi^{n+1}, \nabla(\tilde{\zeta}^{n+1} - \zeta_1^{h,n+1})) = (\tilde{\zeta}^{n+1} - \zeta_1^{h,n+1}, \tilde{\zeta}^{n+1} - \zeta_1^{h,n+1}).
\end{aligned}$$

The first four nonlinear terms in the right hand side of (3.7) can be written as

$$-b^*(\phi^{n+1}, \bar{u}(t_{n+1}); \phi^{n+1}) - b^*(\eta^{n+1}, \bar{u}(t_{n+1}); \phi^{n+1}) - b^*(w_1^{h,n+1}, \eta^{n+1}; \phi^{n+1}).$$

We bound the first nonlinear term using Lemma 2.7 and the generalized Young's inequality:

$$\begin{aligned}
|b^*(\phi^{n+1}, \bar{u}(t_{n+1}); \phi^{n+1})| &\le \|\phi^{n+1}\|^{\frac{1}{2}}\|\nabla\bar{u}(t_{n+1})\|\|\nabla\phi^{n+1}\|^{\frac{3}{2}} \qquad (3.8)\\
&\le \epsilon_1\nu\|\nabla\phi^{n+1}\|^2 + \nu^{-3}\|\nabla\bar{u}(t_{n+1})\|^4\|\phi^{n+1}\|^2,
\end{aligned}$$

for some $\epsilon_1 \in (0,1)$. The other two nonlinear terms are bounded using (2.9):

$$\begin{aligned}
&|b^*(\eta^{n+1}, \bar{u}(t_{n+1}); \phi^{n+1}) + b^*(w_1^{h,n+1}, \eta^{n+1}; \phi^{n+1})| \le \epsilon_2\nu\|\nabla\phi^{n+1}\|^2 \qquad (3.9)\\
&+C\nu^{-1}\|\nabla\bar{u}(t_{n+1})\|^2\|\nabla\eta^{n+1}\|^2 + C\nu^{-1}\|\nabla w_1^{h,n+1}\|^2\|\nabla\eta^{n+1}\|^2
\end{aligned}$$

for some $\epsilon_2 \in (0,1)$.

The last two nonlinear terms of (3.7) are also bounded using (2.9) as follows:

$$\begin{aligned}
&b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1}) - b^*(u(t_{n+1}), u(t_{n+1}); \phi^{n+1}) \qquad (3.10)\\
&= b^*(\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1}) - b^*(u(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1})\\
&+b^*(u(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1}) - b^*(u(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1})\\
&= \delta^2(b^*(\Delta\bar{u}(t_{n+1}), \bar{u}(t_{n+1}); \phi^{n+1}) + b^*(\bar{u}(t_{n+1}), \Delta\bar{u}(t_{n+1}); \phi^{n+1}))\\
&\le \epsilon_2\nu\|\nabla\phi^{n+1}\|^2 + C\nu^{-1}\delta^4\|\nabla^3\bar{u}(t_{n+1})\|^2\|\nabla\bar{u}(t_{n+1})\|^2.
\end{aligned}$$

Plug the bounds (3.8)-(3.10) into (3.7) and use the last equality of (3.7); apply Young's inequality, multiply both sides of (3.7) by $2k$ and sum over the time levels to obtain

$$\|\phi^{n+1}\|_A^2 + 2(1-\epsilon_1-\epsilon_2)\nu k\sum_{i=0}^{n}\left(\|\nabla\phi^{i+1}\|^2 + \delta^2\|\tilde{\zeta}^{n+1} - \zeta_1^{h,n+1}\|^2\right) \tag{3.11}$$
$$\leq Ck\sum_{i=0}^{n}(\|r^{i+1}\|_A^2 + \|\frac{(\bar{u}(t_{i+1}) - \tilde{w}^{i+1}) - (\bar{u}(t_i) - \tilde{w}^i)}{k}\|_A^2$$
$$+\nu^{-1}\|\nabla(\bar{u}(t_{i+1}) - \tilde{w}^{i+1})\|^2 + \nu^{-2}\max_{i=0,1,..,n}\|\nabla(\bar{u}(t_{i+1}) - \tilde{w}^{i+1})\|^2)$$
$$+\nu^{-3}k\sum_{i=0}^{n}\|\nabla\bar{u}(t_{i+1})\|^4\|\phi^{i+1}\|^2,$$

where $r^{i+1} = \frac{\bar{u}(t_{i+1})-\bar{u}(t_i)}{k} - \bar{u}_t(t_{i+1})$. Finally, notice that

$$\|r^{i+1}\|_A^2 \leq Ck^2\|\bar{u}_{tt}(t_{i+1})\|_A^2.$$

Choose $\epsilon_2 = \frac{1-\epsilon_1}{2}$. Use the bound $k < \frac{\nu^3}{\max_{i=0,1,..,N}\|\nabla\bar{u}(t_i)\|^4}$ and apply the discrete Gronwall's lemma 2.8; using the triangle inequality and the error estimate of the Modified Stokes Projection 2.6 completes the proof.
□

3.3. Error estimates for time difference. We now proceed to deriving an error estimate for the time difference $\frac{e_1^{i+1}-e_1^i}{k} = \frac{(\bar{u}(t_{i+1})-w_1^{h,i+1})-(\bar{u}(t_i)-w_1^{h,i})}{k}$, which will be used in the error analysis of the correction step approximation w_2^h. The main result of this subsection is the following

THEOREM 3.3. *Let the conditions of Theorem 3.2 be satisfied. Let also $\bar{u}_t \in L_2(0,T;H_3(\Omega))$ and $\bar{u}_{ttt} \in L_2(0,T;H_1(\Omega))$. Then*

$$\|\frac{e_1^{n+1}-e_1^n}{k}\|_A^2 + \nu k\sum_{i=0}^{n}\|\nabla(\frac{e_1^{n+1}-e_1^n}{k})\|^2 \tag{3.12}$$
$$+\nu\delta^2 k\sum_{i=0}^{n}\|\frac{(\zeta(t_{i+1}) - \zeta_1^{h,i+1}) - (\zeta(t_i) - \zeta_1^{h,i})}{k}\|^2$$
$$\leq C(\nu,\bar{u})(k^2 + \delta^4$$
$$+k\sum_{i=0}^{n}(\inf_{v\in V^h}\|\nabla\frac{(\bar{u}(t_{i+1}) - v^{i+1}) - (\bar{u}(t_i) - v^i)}{k}\|^2$$
$$+\inf_{q\in Q^h}\|\frac{(p(t_{i+1}) - q^{i+1}) - (p(t_i) - q^i)}{k}\|^2))$$

Proof. Consider (3.6) at the $(n+1)$ time level. Choose $v = \frac{\phi^{n+1}-\phi^n}{k} = s^{n+1} \in V_h$, where the error is decomposed as in the previous proof: $e_1^i = \bar{u}(t_i) - w_1^{h,i} = \bar{u}(t_i) - \tilde{w}^i + \tilde{w}^i - w_1^{h,i} = \eta^i + \phi^i$. As in the previous proof, choose $\tilde{w}^{n+1}$ to be the Modified Stokes Projection of $\bar{u}(t_{n+1})$, with $\tilde{\zeta}^{n+1}$ being the Modified Stokes Projection of $\zeta(t_{n+1})$. Take $\xi^h = (\tilde{\zeta}^{n+1} - \zeta_1^{h,n+1}) - (\tilde{\zeta}^n - \zeta_1^{h,n})$. Then consider (3.6) at the previous (n-th) time level and make the same choice $v = s^{n+1} \in V_h$ and $\xi^h = (\tilde{\zeta}^{n+1} - \zeta_1^{h,n+1}) - (\tilde{\zeta}^n - \zeta_1^{h,n}) \in X^h$. Subtracting the two equations and utilizing the definition of Modified Stokes Projection yields:

$$
\begin{aligned}
&(A(s^{n+1}-s^n),s^{n+1})+\nu k(\nabla s^{n+1},\nabla s^{n+1}) \\
&+\nu\delta^2(\nabla((\tilde{\zeta}^{n+1}-\zeta_1^{h,n+1})-(\tilde{\zeta}^n-\zeta_1^{h,n})),\nabla s^{n+1}) \\
&=-(A(\frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{k}),s^{n+1}) \\
&-b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});s^{n+1})+b^*(w_1^{h,n+1},w_1^{h,n+1},s^{n+1}) \\
&+b^*(\bar{u}(t_n),\bar{u}(t_n);s^{n+1})-b^*(w_1^{h,n},w_1^{h,n},s^{n+1}) \\
&+(b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});s^{n+1})-b^*(u(t_{n+1}),u(t_{n+1});s^{n+1})) \\
&-(b^*(\bar{u}(t_n),\bar{u}(t_n);s^{n+1})-b^*(u(t_n),u(t_n);s^{n+1})) \\
&+k(A(\frac{\bar{u}(t_{n+1})-2\bar{u}(t_n)+\bar{u}(t_{n-1})}{k^2}-\frac{\bar{u}_t(t_{n+1})-\bar{u}_t(t_n)}{k}),s^{n+1}), \\
&k(\nabla((\tilde{\zeta}^{n+1}-\zeta_1^{h,n+1})-(\tilde{\zeta}^n-\zeta_1^{h,n})),\nabla s^{n+1}) \\
&=((\tilde{\zeta}^{n+1}-\zeta_1^{h,n+1})-(\tilde{\zeta}^n-\zeta_1^{h,n}),(\tilde{\zeta}^{n+1}-\zeta_1^{h,n+1})-(\tilde{\zeta}^n-\zeta_1^{h,n})).
\end{aligned}
\tag{3.13}
$$

Using the last equality of (3.13), we rewrite (3.13) as

$$
\begin{aligned}
&(A(s^{n+1}-s^n),s^{n+1})+\nu k(\nabla s^{n+1},\nabla s^{n+1}) \\
&+\nu\delta^2 k\|\frac{(\tilde{\zeta}^{n+1}-\zeta_1^{h,n+1})-(\tilde{\zeta}^n-\zeta_1^{h,n})}{k}\|^2 \\
&=-(A(\frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{k}),s^{n+1}) \\
&-b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});s^{n+1})+b^*(w_1^{h,n+1},w_1^{h,n+1},s^{n+1}) \\
&+b^*(\bar{u}(t_n),\bar{u}(t_n);s^{n+1})-b^*(w_1^{h,n},w_1^{h,n},s^{n+1}) \\
&+(b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});s^{n+1})-b^*(u(t_{n+1}),u(t_{n+1});s^{n+1})) \\
&-(b^*(\bar{u}(t_n),\bar{u}(t_n);s^{n+1})-b^*(u(t_n),u(t_n);s^{n+1})) \\
&+k(A(\frac{\bar{u}(t_{n+1})-2\bar{u}(t_n)+\bar{u}(t_{n-1})}{k^2}-\frac{\bar{u}_t(t_{n+1})-\bar{u}_t(t_n)}{k}),s^{n+1}).
\end{aligned}
\tag{3.14}
$$

The first four nonlinear terms in the right hand side of (3.14) can be written as

$$
\begin{aligned}
&-b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});s^{n+1})+b^*(w_1^{h,n+1},w_1^{h,n+1},s^{n+1}) \\
&+b^*(\bar{u}(t_n),\bar{u}(t_n);s^{n+1})-b^*(w_1^{h,n},w_1^{h,n},s^{n+1}) \\
&=-b^*(e_1^{n+1},\bar{u}(t_{n+1});s^{n+1})-b^*(w_1^{h,n+1},e_1^{n+1},s^{n+1}) \\
&+b^*(e_1^n,\bar{u}(t_n);s^{n+1})+b^*(w_1^{h,n},e_1^n,s^{n+1}).
\end{aligned}
\tag{3.15}
$$

Add and subtract $b^*(e_1^n,\bar{u}(t_{n+1});s^{n+1})$ and $b^*(w_1^{h,n+1},e_1^n,s^{n+1})$. We can now rewrite the right hand side of (3.15) as

$$\begin{aligned}
-kb^*(s^{n+1},\bar{u}(t_n);s^{n+1}) - kb^*(\frac{\eta^{n+1}-\eta^n}{k},\bar{u}(t_n);s^{n+1}) \\
-kb^*(e_1^n,\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k};s^{n+1}) \\
-kb^*(w_1^{h,n+1},\frac{\eta^{n+1}-\eta^n}{k};s^{n+1}) \\
+kb^*(\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k},e_1^n,s^{n+1}) - kb^*(\frac{\eta^{n+1}-\eta^n}{k},e_1^n;s^{n+1}) \\
-kb^*(s^{n+1},e_1^n;s^{n+1}).
\end{aligned} \tag{3.16}$$

All the terms in (3.16) are bounded using (2.9), except for the first and the last terms, which require using the result of Lemma 2.7 and the general Young's inequality. In order to simplify the proof, we omit the exact derivation of the bounds; they are obtained in a similar manner to the proof of Theorem 3.2 and require the regularity of the averaged true solution $\bar{u}$ and the accuracy result of Theorem 3.2. However, the last term of 3.16 requires additional attention.

$$\begin{aligned}
kb^*(s^{n+1},e_1^n;s^{n+1}) &\le k\|s^{n+1}\|^{\frac{1}{2}}\|e_1^n\|\|\nabla s^{n+1}\|^{\frac{3}{2}} \\
&\le \epsilon\nu k\|s^{n+1}\|^2 + C\nu^{-3}k\|\nabla e_1^n\|^4\|\nabla s^{n+1}\|^2.
\end{aligned} \tag{3.17}$$

In order to be able to use the discrete Gronwall's lemma later, we need $k\|\nabla e_1^n\|^4 \le O(1)$. We prove this using the result of Theorem 3.2 and the inverse inequality $\|\nabla e_1^n\| \le Ch^{-1}\|\nabla e_1^n\|$. The latter is verified by using the corresponding property of the space X_h (assumption, widely used in the literature) and the accuracy result for the Stokes projection. We obtain from Theorem 3.2 that

$$k\|\nabla e_1^n\|^4 = \frac{(k\|\nabla e_1^n\|^2)^2}{k} \le k^3 + \frac{h^{4r}}{k}, \tag{3.18}$$

where r is the degree of piecewise polynomials in X_h. Also,

$$k\|\nabla e_1^n\|^4 \le Ckh^{-4}\|e_1^n\|^4 \le C(\frac{k^5}{h^4} + \frac{k\delta^8}{h^4} + kh^{4(r-1)}). \tag{3.19}$$

The degree of piecewise polynomials in X_h is strictly positive, since the discrete inf-sup condition is not satisfied otherwise. If $\frac{k}{h^{4r}} \le 1$, use the bound (3.19), otherwise use (3.18). In any case, we obtain that $k\|\nabla e_1^n\|^4 \le C$.

Finally, following (3.10), we bound the last four nonlinear terms of (3.14) as

$$\begin{aligned}
(b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});s^{n+1}) - b^*(u(t_{n+1}),u(t_{n+1});s^{n+1})) \\
-(b^*(\bar{u}(t_n),\bar{u}(t_n);s^{n+1}) - b^*(u(t_n),u(t_n);s^{n+1})) \\
= \delta^2(b^*(\Delta\bar{u}(t_{n+1}),\bar{u}(t_{n+1});s^{n+1}) + b^*(\bar{u}(t_{n+1}),\Delta\bar{u}(t_{n+1});s^{n+1})) \\
-\delta^2(b^*(\Delta\bar{u}(t_n),\bar{u}(t_n);s^{n+1}) - b^*(\bar{u}(t_n),\Delta\bar{u}(t_n);s^{n+1})) \\
\le \epsilon_2\nu k\|\nabla\phi^{n+1}\|^2 \\
+C\nu^{-1}\delta^4k\|\nabla^3\bar{u}(t_{n+1})\|^2\|\nabla(\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k})\|^2 \\
+C\nu^{-1}\delta^4k\|\nabla^3(\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k})\|^2\|\nabla\bar{u}(t_{n+1})\|^2.
\end{aligned} \tag{3.20}$$

Sum (3.14) over the time levels and use the bounds derived for the nonlinear terms. As before, at each time level choose the approximation $\tilde{w}^i$ to be the modified Stokes projection of the averaged solution $\bar{u}(t_i)$ onto X_h. Using the Young's inequality yields

$$\begin{aligned} \|s^{n+1}\|_A^2 + \nu k \sum_{i=0}^{n} \|\nabla s^{i+1}\|^2 \quad (3.21) \\ +\nu\delta^2 k \sum_{i=0}^{n} \|\frac{(\tilde{\zeta}^{i+1} - \zeta_1^{h,i+1}) - (\tilde{\zeta}^i - \zeta_1^{h,i})}{k}\|^2 \\ \leq Ck \sum_{i=0}^{n} (\|r^{i+1}\|_A^2 \\ +\|\frac{(\bar{u}(t_{i+1}) - \tilde{w}^{i+1}) - 2(\bar{u}(t_i) - \tilde{w}^i) + (\bar{u}(t_{i-1}) - \tilde{w}^{i-1})}{k^2}\|_A^2 \\ +\|\nabla e_1^i\|^2) + \nu^{-3} k \sum_{i=0}^{n} \|\nabla e_1^{i+1})\|^4 \|s^{i+1}\|^2, \end{aligned}$$

where $r^{i+1} = \frac{\bar{u}(t_{n+1}) - 2\bar{u}(t_n) + \bar{u}(t_{n-1})}{k^2} - \frac{\bar{u}_t(t_{n+1}) - \bar{u}_t(t_n)}{k}$. Notice also that

$$\|r^{i+1}\|_A^2 \leq Ck^2 \|\bar{u}_{ttt}(t_{i+1})\|_A^2.$$

We now need a bound on $s^1 = \frac{\phi^1 - \phi^0}{k}$. For that, consider (3.6) at the time level $n = 0$. Choose $v = s^1$; also, choosing the initial approximations $w_1^{h,0} = w_2^{h,0}$ to be the modified Stokes projections of $\bar{u}(0)$ onto X_h leads to $\phi^0 = 0$, $e_1^0 = \eta^0$, $\|e_1^0\| \leq Ch^{r+1}$. We also choose $\zeta_1^{h,0}$ to be the Modified Stokes Projection of $\zeta(0)$; take $\xi^h = \tilde{\zeta}^1 - \zeta_1^{h,1}$ and use the definition of the Modified Stokes Projection to obtain

$$\begin{aligned} (As^1, s^1) + \nu(\nabla\phi^1, \nabla s^1) \quad (3.22) \\ +\nu\delta^2(\nabla(\tilde{\zeta}^1 - \zeta_1^{h,1}), \nabla s^1) = -(A(\frac{\eta^1 - \eta^0}{k}), s^1) \\ -b^*(\bar{u}(t_1), \bar{u}(t_1); s^1) + b^*(w_1^{h,1}, w_1^{h,1}, s^1) \\ +(b^*(\bar{u}(t_1), \bar{u}(t_1); v) - b^*(u(t_1), u(t_1); s^1)) \\ +(A(\frac{\bar{u}(t_1) - \bar{u}(t_0)}{k} - \bar{u}_t(t_1)), s^1), \\ k(\nabla s^1, \nabla(\tilde{\zeta}^1 - \zeta_1^{h,1})) = (\tilde{\zeta}^1 - \zeta_1^{h,1}, \tilde{\zeta}^1 - \zeta_1^{h,1}). \end{aligned}$$

Bounding the nonlinear terms as in the general case and using the fact that $\phi^0 = 0$, $\tilde{\zeta}^0 - \zeta_1^{h,0} = 0$, we obtain that

$$\begin{aligned} \|s^1\|_A^2 + \nu k \|\nabla s^1\|^2 + \nu\delta^2 k \|\frac{(\tilde{\zeta}^1 - \zeta_1^{h,1}) - (\tilde{\zeta}^0 - \zeta_1^{h,0})}{k}\|^2 \quad (3.23) \\ \leq \|\frac{\eta^1 - \eta^0}{k}\|_A^2 + \|\frac{\bar{u}(t_1) - \bar{u}(t_0)}{k} - \bar{u}_t(t_1)\|_A^2 + k\|\nabla e_1^1\|^2 + h^{2r}. \end{aligned}$$

Insert this result in (3.21) and use the discrete Gronwall's lemma. Using the triangle inequality completes the proof.
□

4. Stability and accuracy of the second approximation. In this section we investigate the correction step solution w_2^h, satisfying (1.3b). We prove stability and increased accuracy of the solution, therefore deriving a stable method of high temporal and spatial accuracy for solving turbulent Navier-Stokes equations.

4.1. Stability of the second approximation. THEOREM 4.1 (Stability of the second approximation). *Let* w_2^h *satisfy* (1.3b). *Let the conditions of Theorems* 3.1 *and* 3.2 *be satisfied. Then for* $n = 0, ..., N-1$

$$\|w_2^{h,n+1}\|_A^2 + \nu k \Sigma_{i=0}^n \|\nabla w_2^{h,i+1}\|^2 + \nu\delta^2 k \Sigma_{i=0}^n \|\zeta_2^{h,i+1}\|^2 \leq \|w_2^{h,0}\|_A^2 + C\nu^{-2} k \Sigma_{i=0}^n \|\frac{f(t_{i+1}) + f(t_i)}{2}\|_{-1}^2.$$

Proof. Take $v^h = w_2^{h,n+1}$ in (1.3b); let also $\xi^h = \zeta_2^{h,n+1}, j = 2$ in (1.3c) to obtain

$$\begin{aligned}(A(\frac{w_2^{h,n+1} - w_2^{h,n}}{k}), w_2^{h,n+1}) + \nu(\nabla w_2^{h,n+1}, \nabla w_2^{h,n+1}) \quad (4.1)\\ +\nu\delta^2(\nabla\zeta_2^{h,n+1}, \nabla w_2^{h,n+1}) = (\frac{f(t_{n+1}) + f(t_n)}{2}, w_2^{h,n+1})\\ +\frac{\nu}{2}k(\nabla(\frac{w_1^{h,n+1} - w_1^{h,n}}{k}), \nabla w_2^{h,n+1}) + \frac{\nu}{2}\delta^2 k(\nabla(\frac{\zeta_1^{h,n+1} - \zeta_1^{h,n}}{k}), \nabla w_2^{h,n+1})\\ +\frac{1}{2}b^*(w_1^{h,n+1}, w_1^{h,n+1}; w_2^{h,n+1}) - \frac{1}{2}b^*(w_1^{h,n}, w_1^{h,n}; w_2^{h,n+1}),\\ (\nabla w_2^{h,n+1}, \nabla\zeta_2^{h,n+1}) = (\zeta_2^{h,n+1}, \zeta_2^{h,n+1}).\end{aligned}$$

The nonlinear terms in (4.1) are bounded by using (2.9) and Young's inequality:

$$\begin{aligned}\frac{1}{2}b^*(w_1^{h,n+1}, w_1^{h,n+1}; w_2^{h,n+1}) - \frac{1}{2}b^*(w_1^{h,n}, w_1^{h,n}; w_2^{h,n+1}) \quad (4.2)\\ = \frac{1}{2}k(b^*(\frac{w_1^{h,n+1} - w_1^{h,n}}{k}, w_1^{h,n+1}; w_2^{h,n+1}) + b^*(w_1^{h,n}, \frac{w_1^{h,n+1} - w_1^{h,n}}{k}; w_2^{h,n+1}))\\ \leq \epsilon\nu\|\nabla w_2^{h,n+1}\|^2 + C\nu^{-1}k(k\|\nabla w_1^{h,n+1}\|^2\\ +k\|\nabla(\frac{\bar{u}(t_{n+1}) - \bar{u}(t_n)}{k})\|^2 + k\|\nabla(\frac{e_1^{n+1} - e_1^n}{k})\|^2).\end{aligned}$$

Writing

$$\|\nabla(\frac{w_1^{h,n+1} - w_1^{h,n}}{k})\|^2 \leq \|\nabla(\frac{e_1^{n+1} - e_1^n}{k})\|^2 + \|\nabla(\frac{\bar{u}(t_{n+1}) - \bar{u}(t_n)}{k})\|^2,$$

$$\|\nabla(\frac{\zeta_1^{h,n+1} - \zeta_1^{h,n}}{k})\|^2 \leq \|\nabla(\frac{(\zeta(t_{n+1}) - \zeta_1^{h,n+1}) - (\zeta(t_n) - \zeta_1^{h,n})}{k})\|^2 + \|\nabla(\frac{\zeta(t_{n+1}) - \zeta(t_n)}{k})\|^2,$$

and using the result of Theorem 3.3 completes the proof.
$\square$

4.2. Accuracy of the correction step approximation. Rewrite the averaged momentum equation as follows:

$$
\begin{aligned}
&(A(\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k}),v)+\nu(\nabla\bar{u}(t_{n+1}),\nabla v) \\
&+\nu\delta^2(\nabla\zeta(t_{n+1}),\nabla v)+b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});v) \\
&-(\frac{p(t_{n+1})+p(t_n)}{2},\nabla\cdot v)=(\frac{f(t_{n+1})+f(t_n)}{2},v) \\
&+\frac{\nu}{2}k(\nabla(\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k}),\nabla v)+\frac{\nu}{2}\delta^2k(\nabla(\frac{\zeta(t_{n+1})-\zeta(t_n)}{k}),\nabla v) \\
&+\frac{1}{2}(b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});v)-b^*(\bar{u}(t_n),\bar{u}(t_n);v)) \\
&+[\frac{1}{2}(b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});v)+b^*(\bar{u}(t_n),\bar{u}(t_n);v)) \\
&-b^*(u(t_{n+\frac{1}{2}}),u(t_{n+\frac{1}{2}});v)]+(A(\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k}-\bar{u}_t(t_{n+\frac{1}{2}})),v), \\
&(\nabla\bar{u}(t_{n+1}),\nabla\xi^h)=(\zeta(t_{n+1}),\xi^h),\ \forall\xi^h\in X^h.
\end{aligned}
\tag{4.3}
$$

Notice that the last term in the right hand side is second order accurate; compare to the last term in (3.4). The main result of the error analysis in this paper is:

THEOREM 4.2. *Let the assumptions of Theorem* 3.3 *be satisfied. Then the error in the second approximation satisfies*

$$
\begin{aligned}
&\|\bar{u}(t_{n+1})-w_2^{h,n+1}\|_A^2+\nu k\sum_{i=0}^{n}\|\nabla(\bar{u}(t_{i+1})-w_2^{h,i+1})\|^2 \\
&+\nu\delta^2k\sum_{i=0}^{n}\|\zeta(t_{i+1})-\zeta_2^{h,i+1}\|^2 \\
&\leq C(\nu,\bar{u})(k^4+\delta^4+\delta^2k\sum_{i=0}^{n}\inf_{\chi\in X^h}\|\zeta(t_i)-\chi^i\|^2 \\
&+k\sum_{i=0}^{n}(\inf_{v\in V^h}\|\nabla(\bar{u}(t_i)-v^i)\|^2+\inf_{q\in Q^h}\|p(t_i)-q^i\|^2))
\end{aligned}
\tag{4.4}
$$

Proof. Subtract (1.3b) from (4.3) and denote $e_2^i = \bar{u}(t_i) - w_2^{h,i}$ to obtain

$$
\begin{aligned}
&(A(\frac{e_2^{n+1}-e_2^n}{k}),v) + \nu(\nabla e_2^{n+1},\nabla v) \\
&+\nu\delta^2(\nabla(\zeta(t_{n+1})-\zeta_2^{h,n+1}),\nabla v) \\
&+b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});v) - b^*(w_2^{h,n+1},w_2^{h,n+1};v) \\
-(\frac{p(t_{n+1})+p(t_n)}{2} - q_2^{h,n+1},\nabla\cdot v) &= \frac{\nu}{2}k(\nabla(\frac{e_1^{n+1}-e_1^n}{k}),\nabla v) \\
&+\frac{\nu}{2}\delta^2 k(\nabla(\frac{(\zeta(t_{n+1})-\zeta_1^{h,n+1})-(\zeta(t_n)-\zeta_1^{h,n})}{k}),\nabla v) \\
&+\frac{1}{2}(b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});v) - b^*(\bar{u}(t_n),\bar{u}(t_n);v)) \\
&-\frac{1}{2}(b^*(w_1^{h,n+1},w_1^{h,n+1};v) - b^*(w_1^{h,n},w_1^{h,n};v)) \\
+(\frac{1}{2}(b^*(\bar{u}(t_{n+1}),\bar{u}(t_{n+1});v) &+ b^*(\bar{u}(t_n),\bar{u}(t_n);v)) - b^*(u(t_{n+\frac{1}{2}}),u(t_{n+\frac{1}{2}});v)) \\
&+(A(\frac{\bar{u}(t_{n+1})-\bar{u}(t_n)}{k} - \bar{u}_t(t_{n+\frac{1}{2}})),v), \\
(\nabla e_2^{n+1},\nabla\xi^h) &= (\zeta(t_{n+1})-\zeta_2^{h,n+1},\xi^h),\ \forall\xi^h\in X^h.
\end{aligned}
\tag{4.5}
$$

The proof is similar to that of Theorem 3.2. However, the last term of (4.5) will now give the error of the order $O(k^2)$, whereas the corresponding term in (3.6) was first order accurate. There are also the new terms

$$
\frac{\nu}{2}k(\nabla(\frac{e_1^{n+1}-e_1^n}{k}),\nabla v) \text{ and } \frac{\nu}{2}\delta^2 k(\nabla(\frac{(\zeta(t_{n+1})-\zeta_1^{h,n+1})-(\zeta(t_n)-\zeta_1^{h,n})}{k}),\nabla v);
$$

they yield an error of the order

$$
O(k\|\nabla(\frac{e_1^{n+1}-e_1^n}{k})\| + \delta^2 k\|\frac{(\zeta(t_{n+1})-\zeta_1^{h,n+1})-(\zeta(t_n)-\zeta_1^{h,n})}{k}\|).
$$

It is easy to verify that the last three nonlinear terms in (4.5) give the error of the order $O(k^2+\delta^2)$. The other four nonlinear terms are treated similar to the proofs of Theorems 3.2 and 3.3. This leads to

$$
\begin{aligned}
&\|\bar{u}(t_{n+1}) - w_2^{h,n+1}\|_A^2 + \nu k\sum_{i=0}^{n}\|\nabla(\bar{u}(t_{i+1}) - w_2^{h,i+1})\|^2 \\
&+\nu\delta^2 k\sum_{i=0}^{n}\|\zeta(t_{i+1}) - \zeta_2^{h,i+1}\|^2 \\
\leq C(\nu,\bar{u})&(k^2\|\nabla(\frac{e_1^{n+1}-e_1^n}{k})\|^2 + k^2\delta^2\|\frac{(\zeta(t_{n+1})-\zeta_1^{h,n+1})-(\zeta(t_n)-\zeta_1^{h,n})}{k}\|^2 \\
&+k^4+\delta^4+k\sum_{i=0}^{n}(\inf_{v\in V^h}\|\nabla(\bar{u}(t_i)-v^i)\|^2 + \inf_{q\in Q^h}\|p(t_i)-q^i\|^2))
\end{aligned}
\tag{4.6}
$$

Using the result of Theorem 3.3 completes the proof.

$\square$

5. Computational results. In this section we present computational results for the approximate deconvolution model with increased time accuracy. We consider a well-known test problem for the NSE: the two-dimensional wave propagation (considered on a unit square). A Galerkin finite element method is employed, using the Taylor-Hood elements. The results presented were obtained by using the software $FreeFEM++$.

The known solution to the non-averaged NSE in this test problem is

$$u = \begin{pmatrix} 0.75 + 0.25\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2 t/Re} \\ 0.75 - 0.25\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2 t/Re} \end{pmatrix},$$

$$p = -\frac{1}{64}(\cos(4\pi(x-t)) + \cos(4\pi(y-t)))e^{-16\pi^2 t/Re}.$$

We compare the average of this known solution to a solution obtained by our model with the two-dimensional deferred correction and the zeroth order approximate deconvolution employed. It follows from the theoretical results that an error of the order $O((\Delta t)^2 + \delta^2 + h^2)$ is obtained when approximating the averaged true solution $\bar{u}$ by the model solution w. For testing the higher accuracy one would need to employ the finite element spaces of higher degree polynomials.

We take the filtering width $\delta = h$ to verify the acclaimed second order accuracy of the model; this is also a typical choice of filtering widths in real life applications. We also let $\Delta t = h$. The table below demonstrates the first order accuracy of the first approximation and the second order accuracy of the correction step approximation.

TABLE 5.1
Approximating the average solution, $Re = 10^5$, $T = 1$.

h	$\|\bar{u} - w_1\|_{L^2(0,T;L^2(\Omega))}$	$rate$	$\|\bar{u} - w_2\|_{L^2(0,T;L^2(\Omega))}$	$rate$
1/16	0.013047		0.0131178	
1/32	0.00584342	1.16	0.00417758	1.65
1/64	0.00309312	0.92	0.00114869	1.86
1/128	0.00167765	0.88	0.00029897	1.94

Hence, the computational results verify the claimed accuracy of the model. The method proves to be computationally attractive: the number of spacial degrees of freedom is moderate even for $h = 1/128$, and the second order of temporal accuracy is achieved by simply solving the problem twice. Note also, that although the theoretical results were obtained only for the periodic boundary conditions, we have successfully applied the model to the problem with Dirichlet boundary conditions.

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