

2. Note that the net current into buses ④ and ⑤ is

$$\left(\frac{-6.5}{11} + j0\right) + \left(\frac{10.9}{11} + j0\right) = 0.4 + j0$$

3. Note that the net current *specified* into buses ①, ②, ③, ④ and ⑤ is the sum of I_1 , I_2 , I_3 , I_4 and I_5 :

$$(1.5 + j0) + (-0.8 + j0) + (0.5 + j0) + (-1.2 + j0) + (0.4 + j0) = 0.4 + j0$$

Chapter 15 Problem Solutions

- 15.1 The circuit of Fig. 15.1 is redrawn in Fig. 15.7, in which three loop current variables are identified as x_1 , x_2 and x_3 . Although not shown, ammeters and voltmeters with the same accuracy are assumed to be installed as in Fig. 15.1, and the meter readings are also assumed to be the same as those in Example 15.1. Determine the weighted least-squares estimates of the three loop currents. Using the estimated loop currents, determine the source voltages V_1 and V_2 , and compare the results with those of Example 15.1.

Solution:

The meter readings z_1 to z_4 are related to the three loop currents as follows:

$$\begin{aligned} z_1 &= x_1 + e_1 \\ z_2 &= x_2 + e_2 \\ z_3 &= x_1 - x_3 + e_3 \\ z_4 &= x_2 + x_3 + e_4 \end{aligned}$$

The matrix \mathbf{H} is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The inverse of the gain matrix becomes

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.00833 & -0.00167 & 0.00500 \\ -0.00167 & 0.00833 & -0.00500 \\ 0.00500 & -0.00500 & 0.01500 \end{bmatrix}$$

where

$$\mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 50 \end{bmatrix}$$

The estimates of the loop currents are calculated from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 9.01 \\ 3.02 \\ 6.98 \\ 5.01 \end{bmatrix} = \begin{bmatrix} 9.0033 \text{ A} \\ 3.0133 \text{ A} \\ 2.0100 \text{ A} \end{bmatrix}$$

Using the loop currents and elementary circuit analysis, the source voltages are determined to be

$$\begin{aligned} V_1 &= 2\hat{x}_1 - \hat{x}_3 = 15.9966 \text{ V} \\ V_2 &= 2\hat{x}_2 + \hat{x}_3 = 8.0366 \text{ V} \end{aligned}$$

Note that these values are different from those obtained in Example 15.1 because the quantities to be estimated here are the loop currents, while the source voltages were sought before. Another method to find the source voltages is as follows:

$$\begin{aligned} V_1 &= \hat{z}_3 + \hat{z}_1 \times 1 \\ V_2 &= \hat{z}_4 + \hat{z}_2 \times 1 \end{aligned}$$

where \hat{z}_1 to \hat{z}_4 are the estimated measurements such that $\hat{z} = \mathbf{H}\hat{x}$. This will yield the same result.

15.2 Show that $E[(x - \hat{x})(x - \hat{x})^T] = \mathbf{G}^{-1}$ where \mathbf{G} is the gain matrix.

Solution:

$$\begin{aligned} E[(x - \hat{x})(x - \hat{x})^T] &= E[(\mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{e})(\mathbf{e}^T\mathbf{W}\mathbf{H}\mathbf{G}^{-1T})] \\ &= \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{E}[\mathbf{e}\mathbf{e}^T]\mathbf{W}\mathbf{H}\mathbf{G}^{-1T} \\ &= \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{W}^{-1}\mathbf{W}\mathbf{H}\mathbf{G}^{-1T} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{H}\mathbf{G}^{-1T} \\ &= \mathbf{G}^{-1}\mathbf{G}\mathbf{G}^{-1T} = \mathbf{G}^{-1} \end{aligned}$$

15.3 Show that the sum of the diagonal elements in the matrix $\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}$ in Eq. (15.40) is numerically equal to the number of state variables.

Solution:

The sum of the diagonal elements of $\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}$ is

$$\text{trace} [\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}] = \text{trace} \left[\underbrace{\mathbf{H}(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}}_{\mathbf{X}} \underbrace{\mathbf{H}^T\mathbf{R}^{-1}}_{\mathbf{Y}} \right]$$

since the trace $(\mathbf{X}\mathbf{Y}) = \text{trace}(\mathbf{Y}\mathbf{X})$, when such products are meaningful,

$$\begin{aligned} \text{trace} [\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}] &= \text{trace} \left[\underbrace{\mathbf{H}^T\mathbf{R}^{-1}}_{\mathbf{Y}} \underbrace{\mathbf{H}(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}}_{\mathbf{X}} \right] \\ &= \text{trace} [(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}] = \text{trace} [\mathbf{I}] = N_s \end{aligned}$$

Note that matrix \mathbf{I} size is $N_s \times N_s$.

15.4 Prove Eq. (15.47).

Solution:

$$\begin{aligned}
 \hat{f} &= \hat{e}^T R^{-1} \hat{e} \\
 &= \{ [I - HG^{-1}H^T R^{-1}] e \}^T R^{-1} [I - HG^{-1}H^T R^{-1}] e \\
 &= e^T \{ R^{-1} - R^{-1T} H G^{-1T} H^T R^{-1} \} [I - HG^{-1}H^T R^{-1}] e \\
 &= e^T R^{-1} [I - HG^{-1}H^T R^{-1}] [I - HG^{-1}H^T R^{-1}] e \\
 &= e^T R^{-1} [I - HG^{-1}H^T R^{-1}] e \\
 &\quad \text{(The matrix in the bracket is an idempotent matrix)}
 \end{aligned}$$

Note that

$$E[e_i^2] = \sigma_i^2 = \frac{1}{R_{ii}}$$

Since measurements i and j are uncorrelated, $E[\hat{e}_i \hat{e}_j] = 0$. Therefore,

$$\begin{aligned}
 E[\hat{f}] &= \text{trace} [I - HG^{-1}H^T R^{-1}] \\
 &= \text{trace} [I] - \text{trace} [HG^{-1}H^T R^{-1}] \\
 &= N_m - \text{trace} \left\{ [H (H^T R^{-1} H)^{-1}] [H^T R^{-1}] \right\} \\
 &= N_m - \text{trace} \left\{ [H^T R^{-1}] [H (H^T R^{-1} H)^{-1}] \right\} = N_m - N_s
 \end{aligned}$$

15.5 Consider the voltages at the two nodes labeled ① and ② in the circuit of Fig. 15.7 to be state variables. Using the ammeters and voltmeters connected as shown in Fig. 15.1 and their readings given in Example 15.1, determine the weighted least-squares estimates of these node voltages. Using the result, determine the source voltages V_1 and V_2 , and compare the results with those of Example 15.1. Also calculate the expected value of the sum-of-squares of the measurement residuals using Eq. (15.46), and check your answer using Eq. (15.47).

Solution:

The meter readings are related to the node voltages x_1 and x_2 by

$$\begin{aligned}
 z_1 &= x_1 + x_1 - x_2 + e_1 = 2x_1 - x_2 + e_1 \\
 z_2 &= x_2 + x_2 - x_1 + e_2 = -x_1 + 2x_2 + e_2 \\
 z_3 &= x_1 + e_3 \\
 z_4 &= x_2 + e_4
 \end{aligned}$$

The matrix H is

$$H = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.00386 & 0.00281 \\ 0.00281 & 0.00386 \end{bmatrix}$$

The estimates of the node voltages are found as

$$\begin{aligned} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} &= \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} \\ &= \mathbf{G}^{-1} \mathbf{H}^T \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} 9.01 \\ 3.02 \\ 6.98 \\ 5.01 \end{bmatrix} = \begin{bmatrix} 7.0060 \text{ V} \\ 5.0107 \text{ V} \end{bmatrix} \end{aligned}$$

Using the node voltages found, the source voltages are

$$\begin{aligned} V_1 &= \hat{x}_1 + \hat{z}_1 = 7.0060 + 9.0012 = 16.0072 \text{ V} \\ V_2 &= \hat{x}_2 + \hat{z}_2 = 5.0107 + 3.0154 = 8.0261 \text{ V} \end{aligned}$$

where $\hat{\mathbf{z}} = \mathbf{H}\hat{\mathbf{x}}$. It is seen that the estimated source voltages are the same as those of Example 15.1. The expected value of the sum of squares of the measurement residuals is

$$\begin{aligned} E[\hat{f}] &= \sum_{j=1}^{N_m} \frac{R'_{jj}}{\sigma_j^2} \\ &= \frac{0.00193}{(0.1)^2} + \frac{0.00193}{(0.1)^2} + \frac{0.01614}{(\sqrt{2}/10)^2} + \frac{0.01614}{(\sqrt{2}/10)^2} = 2 \end{aligned}$$

which can be checked by

$$N_m - N_s = 4 - 2 = 2 \quad \checkmark$$

15.6 Five ammeters numbered A_1 to A_5 are used in the dc circuit of Fig. 15.8 to determine the two unknown source currents I_1 and I_2 . The standard deviations of the meter errors are 0.2 A for meters A_2 and A_5 , and 0.1 A for the other three meters. The readings of the five meters are 0.12 A, 1.18 A, 3.7 A, 0.81 A and 7.1 A, respectively.

- Determine the weighted least-squares estimates of the source currents I_1 and I_2 .
- Using the chi-square test of Eq. (15.49) for $\alpha = 0.01$, check for the presence of bad data in the measurements.
- Eliminate any bad data detected in (b) and find the weighted least-squares estimates of the source currents using the reduced data set.
- Apply the chi-square test for $\alpha = 0.01$ to the results of (c) to check if the result is statistically acceptable.

Solution:

- (a) Let the unknown source currents be denoted by x_1 and x_2 , and the meter readings by z_1 through z_5 . It follows from circuit analysis that

$$\begin{aligned} z_1 &= \frac{7}{40}x_1 - \frac{3}{40}x_2 + e_1 \\ z_2 &= \frac{-3}{40}x_1 + \frac{7}{40}x_2 + e_2 \\ z_3 &= \frac{33}{40}x_1 + \frac{3}{40}x_2 + e_3 \\ z_4 &= \frac{4}{40}x_1 + \frac{4}{40}x_2 + e_4 \\ z_5 &= \frac{3}{40}x_1 + \frac{33}{40}x_2 + e_5 \end{aligned}$$

from which the matrix \mathbf{H} is

$$\mathbf{H} = \begin{bmatrix} 0.175 & -0.075 \\ -0.075 & 0.175 \\ 0.825 & 0.075 \\ 0.100 & 0.100 \\ 0.075 & 0.825 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.01431 & -0.00510 \\ -0.00510 & 0.05205 \end{bmatrix}$$

where

$$\mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{bmatrix}$$

The estimates of the source currents are determined to be

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} = \begin{bmatrix} 3.7218 \text{ A} \\ 8.0451 \text{ A} \end{bmatrix}$$

(b)

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} - \begin{bmatrix} 0.04793 \\ 1.12877 \\ 3.67385 \\ 1.17669 \\ 6.91638 \end{bmatrix} = \begin{bmatrix} 0.07207 \\ 0.05123 \\ 0.02615 \\ -0.36669 \\ 0.18362 \end{bmatrix}$$

$$\begin{aligned} \hat{J} &= \sum_{j=1}^5 \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 100 \times 0.07207^2 + 25 \times 0.05123^2 + 100 \times 0.02615^2 + 100 \times (-0.36669)^2 + 25 \times 0.18362^2 \\ &= 14.9427 \end{aligned}$$

Since $N_m = 5$ and $N_s = 2$, $k = N_m - N_s = 3$. For $\alpha = 0.01$, $\chi_{3,0.01}^2 = 11.35$. There is at least one bad measurement by observing that $\hat{f} > 11.35$.

- (c) To find the standardized error-estimates, the diagonal elements of the matrix \mathbf{R}' are first calculated using \mathbf{H} and \mathbf{G}^{-1} obtained above. Thus,

$$\begin{aligned}\mathbf{R}' &= (\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1})\mathbf{R} \\ &= \begin{bmatrix} 0.00914 & \times & \times & \times & \times \\ \times & 0.03819 & \times & \times & \times \\ \times & \times & 0.00060 & \times & \times \\ \times & \times & \times & 0.00944 & \times \\ \times & \times & \times & \times & 0.00512 \end{bmatrix} \\ &\text{where } \mathbf{R} = \mathbf{W}^{-1}\end{aligned}$$

The standardized errors are calculated from

$$\begin{aligned}\frac{\hat{e}_1}{\sqrt{R'_{11}}} &= \frac{0.07207}{\sqrt{0.00914}} = 0.75410 \\ \frac{\hat{e}_2}{\sqrt{R'_{22}}} &= \frac{0.05123}{\sqrt{0.03819}} = 0.26216 \\ \frac{\hat{e}_3}{\sqrt{R'_{33}}} &= \frac{0.02615}{\sqrt{0.00060}} = 1.06923 \\ \frac{\hat{e}_4}{\sqrt{R'_{44}}} &= \frac{-0.36669}{\sqrt{0.00944}} = -3.77445 \\ \frac{\hat{e}_5}{\sqrt{R'_{55}}} &= \frac{0.18362}{\sqrt{0.00512}} = 2.56567\end{aligned}$$

from which z_4 is identified as the bad measurement. To perform state estimation without z_4 , the 4th row of \mathbf{H} is eliminated and the 4th row and 4th column of \mathbf{W} are also eliminated. The inverse of the new gain matrix becomes

$$\mathbf{G}^{-1} = (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1} = \begin{bmatrix} 0.01440 & -0.00464 \\ -0.00464 & 0.05439 \end{bmatrix}$$

where

$$\mathbf{H} = \begin{bmatrix} 0.175 & -0.075 \\ -0.075 & 0.175 \\ 0.825 & 0.075 \\ 0.075 & 0.825 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$$

The new estimates of the source currents are then given by

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 7.1 \end{bmatrix} = \begin{bmatrix} 3.75756 \text{ A} \\ 8.22756 \text{ A} \end{bmatrix}$$

(d)

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 7.1 \end{bmatrix} - \begin{bmatrix} 0.04051 \\ 1.15801 \\ 3.71706 \\ 7.06956 \end{bmatrix} = \begin{bmatrix} 0.07949 \\ 0.02199 \\ -0.01706 \\ 0.03044 \end{bmatrix}$$

$$\begin{aligned}
 \hat{f} &= \sum_{\substack{j=1 \\ j \neq 4}}^5 \frac{\hat{e}_j^2}{\sigma_j^2} \\
 &= 100 \times 0.07949^2 + 25 \times 0.02199^2 + 100 \times (-0.01706)^2 + 25 \times 0.03044^2 \\
 &= 0.69628
 \end{aligned}$$

Note that in this case $k = N_m - N_s = 4 - 2 = 2$. For $\alpha = 0.01$, $\chi_{2,0.01}^2 = 9.21$. Since $\hat{f} < 9.21$, it is concluded that the set of remaining four measurements does not have any bad measurements.

15.7 Re-do Prob. 15.6 when the unknowns to be determined are not the source currents, but the voltages at the three nodes labeled ①, ② and ③ in Fig. 15.8.

Solution:

Let the voltages at the three nodes identified as ①, ② and ③ in Fig. 15.8 be denoted by x_1 , x_2 and x_3 , respectively. From circuit analysis,

$$\begin{aligned}
 z_1 &= (x_1 - x_2)/3 + e_1 \\
 z_2 &= (x_3 - x_2)/3 + e_2 \\
 z_3 &= x_1 + e_3 \\
 z_4 &= x_2/3 + e_4 \\
 z_5 &= x_3/1 + e_5
 \end{aligned}$$

from which matrix \mathbf{H} is

$$\mathbf{H} = \begin{bmatrix} 1/3 & -1/3 & 0 \\ 0 & -1/3 & 1/3 \\ 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.00942 & 0.00424 & 0.00042 \\ 0.00424 & 0.04235 & 0.00424 \\ 0.00042 & 0.00424 & 0.03642 \end{bmatrix}$$

where

$$\mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{bmatrix}$$

The estimates of the node voltages are determined to be

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} = \begin{bmatrix} 3.65951 \text{ V} \\ 2.93507 \text{ V} \\ 7.03751 \text{ V} \end{bmatrix}$$

To check for the presence of bad data,

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} - \begin{bmatrix} 0.24148 \\ 1.36747 \\ 3.65951 \\ 0.97835 \\ 7.03751 \end{bmatrix} = \begin{bmatrix} -0.12148 \\ -0.18747 \\ 0.04049 \\ -0.16835 \\ 0.06249 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^5 \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 100 \times (-0.12148)^2 + 25 \times (-0.18747)^2 + 100 \times 0.04049^2 + 100 \times (-0.16835)^2 + 25 \times 0.06249^2 \\ &= 5.44991 \end{aligned}$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi_{2,0.01}^2 = 9.21$. Since $\hat{f} < 9.21$, the set of measurements has no bad data for the specified confidence level.

15.8 Consider the circuit of Fig. 15.8 for which accuracy of the ammeters and their readings are the same as those specified in Prob. 15.6. As in Prob. 15.7, the voltages at the three nodes labeled ①, ② and ③ are to be estimated without first finding the source currents.

- (a) Suppose that meters A_4 and A_5 are found to be out of order, and therefore, only three measurements $z_1 = 0.12$, $z_2 = 1.18$ and $z_3 = 3.7$ are available. Determine the weighted least-squares estimates of the nodal voltages, and the estimated errors \hat{e}_1 , \hat{e}_2 and \hat{e}_3 .
- (b) This time suppose meters A_2 and A_5 are now out of order and the remaining three meters are working. Using three measurements $z_1 = 0.12$, $z_3 = 3.7$ and $z_4 = 0.81$, can the nodal voltages be estimated without finding the source currents first? Explain why by examining the matrix \mathbf{G} .

Solution:

- (a) By eliminating the 4th and 5th rows from \mathbf{H} obtained in Prob. 15.7, we have

$$\mathbf{H} = \begin{bmatrix} 1/3 & -1/3 & 0 \\ 0 & -1/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.10 & 0.10 \\ 0.01 & 0.10 & 0.46 \end{bmatrix}$$

The estimates of the node voltages are determined to be

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \end{bmatrix} = \begin{bmatrix} 3.7 \text{ V} \\ 3.34 \text{ V} \\ 6.88 \text{ V} \end{bmatrix}$$

It is easy to check that

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \end{bmatrix} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the degree of freedom is zero. Therefore, in this case, the states of the system are completely defined without any redundancy in measurements.

- (b) If only z_1 , z_3 and z_4 are used, node ③ cannot be represented at all and, consequently, its nodal voltage cannot be estimated. This can also be checked by examining the matrix \mathbf{H} for this case:

$$\mathbf{H} = \begin{bmatrix} 1/3 & -1/3 & 0 \\ 1 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

Note that all the elements in the 3rd column are 0, indicating that x_3 (the voltage at node ③) cannot affect the measurements. One can also note that the gain matrix becomes

$$\begin{aligned} \mathbf{G} = \mathbf{H}^T \mathbf{W} \mathbf{H} &= \begin{bmatrix} 1/3 & 1 & 0 \\ -1/3 & 0 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 & 0 \\ 1 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -100/9 & 0 \\ -100/9 & 200/9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

which cannot be inverted.

15.9 Suppose that the two voltage sources in Example 15.1 have been replaced with new ones, and the meter readings now show $z_1 = 2.9$ A, $z_2 = 10.2$ A, $z_3 = 5.1$ V and $z_4 = 7.2$ V.

- Determine the weighted least-squares estimates of the new source voltages.
- Using the chi-square test for $\alpha = 0.005$, detect bad data.
- Eliminate the bad data and determine again the weighted least-squares estimates of the source voltages.
- Check your result in (c) again using the chi-square test.

Solution:

- (a) The estimates of the new source voltages are obtained from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 2.9 \\ 10.2 \\ 5.1 \\ 7.2 \end{bmatrix} = \begin{bmatrix} 8.00175 \text{ V} \\ 17.66491 \text{ V} \end{bmatrix}$$

where \mathbf{G} , \mathbf{H} and \mathbf{W} are all specified in Example 15.1.

(b) To verify the presence of bad data,

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 2.9 \\ 10.2 \\ 5.1 \\ 7.2 \end{bmatrix} - \begin{bmatrix} 2.79298 \\ 10.04035 \\ 5.20877 \\ 7.62456 \end{bmatrix} = \begin{bmatrix} 0.10702 \\ 0.15965 \\ -0.10877 \\ -0.42456 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^4 \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 100 \times 0.10702^2 + 100 \times 0.15965^2 + 50 \times (-0.10877)^2 + 50 \times (-0.42456)^2 \\ &= 13.29826 \end{aligned}$$

Note that in this case $k = N_m - N_s = 4 - 2 = 2$. For $\alpha = 0.005$, $\chi_{2,0.005}^2 = 10.60$. Since $\hat{f} > 10.60$, there is at least one bad measurement.

(c) The diagonal elements of matrix \mathbf{R}' are first found as

$$\begin{aligned} \mathbf{R}' &= (\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1})\mathbf{R} \\ &= \begin{bmatrix} 0.00193 & \times & \times & \times \\ \times & 0.00193 & \times & \times \\ \times & \times & 0.01614 & \times \\ \times & \times & \times & 0.01614 \end{bmatrix} \\ &\text{where } \mathbf{R} = \mathbf{W}^{-1} \end{aligned}$$

The standardized errors are calculated from

$$\begin{aligned} \frac{\hat{e}_1}{\sqrt{R'_{11}}} &= \frac{0.10702}{\sqrt{0.00193}} = 2.43611 \\ \frac{\hat{e}_2}{\sqrt{R'_{22}}} &= \frac{0.15965}{\sqrt{0.00193}} = 3.63421 \\ \frac{\hat{e}_3}{\sqrt{R'_{33}}} &= \frac{-0.10877}{\sqrt{0.01614}} = -0.85617 \\ \frac{\hat{e}_4}{\sqrt{R'_{44}}} &= \frac{-0.42456}{\sqrt{0.01614}} = -3.34183 \end{aligned}$$

from which z_2 is identified as the bad measurement. To perform state estimation without z_2 , the 2nd row of \mathbf{H} is eliminated and the 2nd row and 2nd column of \mathbf{W} are also eliminated. The inverse of the new gain matrix becomes

$$\mathbf{G}^{-1} = (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1} = \begin{bmatrix} 0.02182 & 0.00727 \\ 0.00727 & 0.10909 \end{bmatrix}$$

where

$$\mathbf{H} = \begin{bmatrix} 0.625 & -0.125 \\ 0.375 & 0.125 \\ 0.125 & 0.375 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

The new estimates of the source voltages are then given by

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 2.9 \\ 5.1 \\ 7.2 \end{bmatrix} = \begin{bmatrix} 7.97273 \text{ A} \\ 16.59091 \text{ A} \end{bmatrix}$$

(d)

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 2.9 \\ 5.1 \\ 7.2 \end{bmatrix} - \begin{bmatrix} 2.90909 \\ 5.06364 \\ 7.21818 \end{bmatrix} = \begin{bmatrix} -0.00909 \\ 0.03636 \\ -0.01818 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^3 \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 100 \times (-0.00909)^2 + 50 \times 0.03636^2 + 50 \times (-0.01818)^2 \\ &= 0.09091 \end{aligned}$$

Note that $k = N_m - N_s = 3 - 2 = 1$. For $\alpha = 0.005$, $\chi_{1,0.005}^2 = 7.88$. Since $\hat{f} < 7.88$, no more bad data exists.

15.10 Five wattmeters are installed on the four-bus system of Fig. 15.9 to measure line real power flows, where per unit reactances of the lines are $X_{12} = 0.05$, $X_{13} = 0.1$, $X_{23} = 0.04$, $X_{24} = 0.0625$ and $X_{34} = 0.08$. Suppose that the meter readings show that

$$\begin{aligned} z_1 &= P_{12} = 0.34 \text{ per unit} \\ z_2 &= P_{13} = 0.26 \text{ per unit} \\ z_3 &= P_{23} = 0.17 \text{ per unit} \\ z_4 &= P_{24} = -0.24 \text{ per unit} \\ z_5 &= P_{34} = -0.22 \text{ per unit} \end{aligned}$$

where the variances of the measurement errors in per unit are given by

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = (0.01)^2$$

- Apply the dc power-flow method of Sec. 9.7 to this network with bus ① as reference, and determine the corresponding \mathbf{H} matrix. Then, compute the weighted least-squares estimates of the phase angles of the bus voltages in radians.
- Using the chi-square test for $\alpha = 0.01$, identify two bad measurements. Between the two bad measurements, one is not worse than the other as far as accuracy is concerned. Explain why. If both bad measurements are eliminated simultaneously, would it be possible to estimate the states of the system?
- For the two bad measurements identified in (b), determine the relationship between the two error estimates in terms of the reactances of the corresponding two lines.

- (d) Eliminate one of the bad measurements identified in (b), and determine the weighted least-squares estimates of the phase angles of the bus voltages using the reduced data set. Do the same for the other bad measurement. By comparing the two results, identify the buses at which the estimated phase angles are equal in the two cases.

Solution:

- (a) Let x_1 , x_2 and x_3 represent the phase angles of the bus voltages at buses ②, ③ and ④, respectively. With bus ① as the reference (with the phase angle of 0 radians), a dc power flow analysis would yield

$$\begin{aligned} z_1 &= \frac{1}{0.05} (0 - x_1) + e_1 \\ z_2 &= \frac{1}{0.1} (0 - x_2) + e_2 \\ z_3 &= \frac{1}{0.04} (x_1 - x_2) + e_3 \\ z_4 &= \frac{1}{0.0625} (x_1 - x_3) + e_4 \\ z_5 &= \frac{1}{0.08} (x_2 - x_3) + e_5 \end{aligned}$$

from which matrix \mathbf{H} is

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.20499 & 0.18005 & 0.19554 \\ 0.18005 & 0.27980 & 0.21786 \\ 0.19554 & 0.21786 & 0.44657 \end{bmatrix} \times 10^{-6}$$

where

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 & 0 \\ 0 & 0 & 10000 & 0 & 0 \\ 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 10000 \end{bmatrix}$$

The estimates of the phase angles (in radians) of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \\ -0.22 \end{bmatrix} = \begin{bmatrix} -0.01750 \\ -0.02401 \\ -0.00398 \end{bmatrix}$$

(b)

$$\hat{e} = z - \hat{z} = z - H\hat{x} = \begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \\ -0.22 \end{bmatrix} - \begin{bmatrix} 0.34996 \\ 0.24009 \\ 0.16278 \\ -0.21328 \\ -0.25036 \end{bmatrix} = \begin{bmatrix} -0.00996 \\ 0.01991 \\ 0.00722 \\ -0.02372 \\ 0.03036 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^5 \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 10000 \times [(-0.00996)^2 + 0.01991^2 + 0.00722^2 + (-0.02372)^2 + 0.03036^2] \\ &= 20.31951 \end{aligned}$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi_{2,0.01}^2 = 9.21$. Since $\hat{f} > 9.21$, there exists at least one bad measurement. To find the bad measurement, the diagonal elements of R' are computed:

$$\begin{aligned} R' &= (I - HG^{-1}H^TR^{-1})R \\ &= \begin{bmatrix} 0.18005 & \times & \times & \times & \times \\ \times & 0.72020 & \times & \times & \times \\ \times & \times & 0.22073 & \times & \times \\ \times & \times & \times & 0.33317 & \times \\ \times & \times & \times & \times & 0.54586 \end{bmatrix} \times 10^{-4} \end{aligned}$$

The standardized errors are calculated from

$$\begin{aligned} \frac{\hat{e}_1}{\sqrt{R'_{11}}} &= \frac{-0.00996}{\sqrt{0.18005 \times 10^{-4}}} = -2.34601 \\ \frac{\hat{e}_2}{\sqrt{R'_{22}}} &= \frac{0.01991}{\sqrt{0.72020 \times 10^{-4}}} = 2.34601 \\ \frac{\hat{e}_3}{\sqrt{R'_{33}}} &= \frac{0.00722}{\sqrt{0.22073 \times 10^{-4}}} = 1.53608 \\ \frac{\hat{e}_4}{\sqrt{R'_{44}}} &= \frac{-0.02372}{\sqrt{0.33317 \times 10^{-4}}} = -4.10936 \\ \frac{\hat{e}_5}{\sqrt{R'_{55}}} &= \frac{0.03036}{\sqrt{0.54586 \times 10^{-4}}} = 4.10936 \end{aligned}$$

Note that the standardized errors for the 4th and 5th measurements are equally bad. This can be expected since both measurements z_4 and z_5 are equally affected by the voltage phase angle at bus ④. Note that the standardized errors for the first and second measurements are also equal. If both z_4 and z_5 are discarded, bus ④ will virtually be disconnected from the system, making state estimation impossible. The elimination of both z_4 and z_5 is also equivalent to deleting the 4th and 5th rows from H . It is easy to check that the resulting gain matrix cannot be inverted.

(c) From (b) above, we have

$$\begin{aligned} \hat{e}_4 &= z_4 - \hat{z}_4 = -0.02372 \\ \hat{e}_5 &= z_5 - \hat{z}_5 = 0.03036 \end{aligned}$$

Note that in the equations relating x to z , the absolute magnitude of the coefficients of x_3 in the z_4 and z_5 equations are $1/0.0625$ and $1/0.08$, respectively, as can be seen in \mathbf{H} . This means that z_4 and z_5 are sensitive to changes in x_3 in the ratio of $0.08:0.0625$. Consequently, the error estimates will be inversely proportional to this ratio ($0.0625:0.08$). This can be verified by observing that

$$\frac{|\hat{e}_4|}{|\hat{e}_5|} = \frac{0.02372}{0.03036} = \frac{0.0625}{0.08} = \frac{X_{24}}{X_{34}}$$

(d) By deleting z_4 , the reduced matrix \mathbf{H} becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The inverse of the new gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.17731 \\ 0.17731 & 0.29078 & 0.29078 \\ 0.17731 & 0.29078 & 0.93078 \end{bmatrix} \times 10^{-6}$$

where

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 \\ 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 10000 \end{bmatrix}$$

The new estimates of the phase angles of the bus voltages, in the absence of z_4 , are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.22 \end{bmatrix} = \begin{bmatrix} -0.01739 \\ -0.02444 \\ -0.00684 \end{bmatrix}$$

If z_5 is deleted instead, the reduced matrix \mathbf{H} becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \end{bmatrix}$$

In this case, the inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.20567 \\ 0.17731 & 0.29078 & 0.17731 \\ 0.20567 & 0.17731 & 0.59630 \end{bmatrix} \times 10^{-6}$$

In this case, the new estimates of the phase angles of the bus voltages are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \end{bmatrix} = \begin{bmatrix} -0.01739 \\ -0.02444 \\ -0.00239 \end{bmatrix}$$

From the above two results, the phase angles are equal for the two cases at buses ①, ② and ③, respectively. This could be expected because the state estimation procedure, in the absence of either z_4 or z_5 , will minimize \hat{f} with respect to the line measurements connecting buses ①, ② and ③. Note that the phase angle at bus ④ becomes a dependent state variable if no measurement is made on either line ②-④ or line ③-④.

- 15.11 In the four-bus system of Prob. 15.10, suppose that the variance of the measurement error for z_5 is $(0.05)^2$ and that all the other data remain the same. Qualitatively describe how the newly estimated values z_4 and z_5 of the measurements will differ from those obtained in Prob. 15.10. Verify your answer by recalculating the weighted least-squares estimates of the phase angles (in radians) of the bus voltages and the corresponding \hat{z} .

Solution:

Since $\sigma_5^2 = 0.05^2 > \sigma_4^2 = 0.01^2$, \hat{x}_3 will be determined such that the corresponding \hat{z}_4 is much closer to the measurement z_4 than \hat{z}_5 is to z_5 . The \mathbf{H} matrix is the same as that found in Prob. 15.10.

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

However, \mathbf{W} is different and is given by

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 & 0 \\ 0 & 0 & 10000 & 0 & 0 \\ 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 400 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.20563 & 0.17750 & 0.20496 \\ 0.17750 & 0.29000 & 0.18018 \\ 0.20496 & 0.18018 & 0.58568 \end{bmatrix} \times 10^{-6}$$

The estimates of the phase angles of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \\ -0.22 \end{bmatrix} = \begin{bmatrix} -0.01740 \\ -0.02441 \\ -0.00250 \end{bmatrix}$$

The estimated values of z_i 's are calculated from

$$\hat{\mathbf{z}} = \mathbf{H} \hat{\mathbf{x}} = \begin{bmatrix} 0.34795 \\ 0.24409 \\ 0.17529 \\ -0.23832 \\ -0.27383 \end{bmatrix}$$

For the fourth and fifth measurements,

$$\begin{aligned}\hat{e}_4 &= z_4 - \hat{z}_4 = -0.24 - (-0.23832) = -0.00168 \\ \hat{e}_5 &= z_5 - \hat{z}_5 = -0.22 - (-0.27383) = 0.05383\end{aligned}$$

Note that the weighted least squares estimation procedure regards z_4 much more accurately than z_5 .

- 15.12 Suppose that a line of impedance $j0.025$ per unit is added between buses ① and ④ in the network of Fig. 15.9, and that a wattmeter is installed on this line at bus ①. The variance of the measurement error for this added wattmeter is assumed to be the same as that of the others. The meter readings now show

$$\begin{aligned}z_1 &= P_{12} = 0.32 \text{ per unit} \\ z_2 &= P_{13} = 0.24 \text{ per unit} \\ z_3 &= P_{23} = 0.16 \text{ per unit} \\ z_4 &= P_{24} = -0.29 \text{ per unit} \\ z_5 &= P_{34} = -0.27 \text{ per unit} \\ z_6 &= P_{14} = 0.05 \text{ per unit}\end{aligned}$$

- (a) Find the \mathbf{H} matrix that describes the dc power flow with bus ① as reference, and compute the weighted least-squares estimates of the phase angles of the bus voltages in radians.
- (b) Using the chi-square test for $\alpha = 0.01$, eliminate any bad data and recompute the weighted least-squares estimates of the phase angles of the bus voltages. Check your result again using the chi-square test for $\alpha = 0.01$.

Solution:

- (a) Let x_1 , x_2 and x_3 denote the phase angles of the voltages at buses ②, ③ and ④, respectively. With the phase angle of the voltage at bus ① specified to be 0, dc power flow analysis provides

$$\begin{aligned}z_1 &= \frac{1}{0.05} (0 - x_1) + e_1 \\ z_2 &= \frac{1}{0.1} (0 - x_2) + e_2 \\ z_3 &= \frac{1}{0.04} (x_1 - x_2) + e_3 \\ z_4 &= \frac{1}{0.0625} (x_1 - x_3) + e_4 \\ z_5 &= \frac{1}{0.08} (x_2 - x_3) + e_5 \\ z_6 &= \frac{1}{0.025} (0 - x_3) + e_6\end{aligned}$$

from which matrix \mathbf{H} is

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \\ 0 & 0 & -40 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.12988 & 0.09637 & 0.02401 \\ 0.09637 & 0.18657 & 0.02675 \\ 0.02401 & 0.02675 & 0.05483 \end{bmatrix} \times 10^{-6}$$

where

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10000 \end{bmatrix}$$

The estimates of the phase angles (in radians) of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \\ 0.05 \end{bmatrix} = \begin{bmatrix} -0.01712 \\ -0.02341 \\ -0.00101 \end{bmatrix}$$

(b)

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H} \hat{\mathbf{x}} = \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \\ 0.05 \end{bmatrix} - \begin{bmatrix} 0.34240 \\ 0.23412 \\ 0.15732 \\ -0.25781 \\ -0.28007 \\ 0.04027 \end{bmatrix} = \begin{bmatrix} -0.02240 \\ 0.00588 \\ 0.00268 \\ -0.03219 \\ 0.01007 \\ 0.00973 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^6 \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 10000 \times [(-0.02240)^2 + 0.00588^2 + 0.00268^2 + (-0.03219)^2 + 0.01007^2 + 0.00973^2] \\ &= 17.7568 \end{aligned}$$

Note that $k = N_m - N_s = 6 - 3 = 3$. For $\alpha = 0.01$, $\chi_{3,0.01}^2 = 11.35$. Since $\hat{f} > 11.35$, there exists at least one bad measurement. To identify the bad measurement, the diagonal elements of matrix \mathbf{R}' are computed from

$$\mathbf{R}' = (\mathbf{I} - \mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1}) \mathbf{R}$$

$$= \begin{bmatrix} 0.48048 & \times & \times & \times & \times & \times \\ \times & 0.81344 & \times & \times & \times & \times \\ \times & \times & 0.22684 & \times & \times & \times \\ \times & \times & \times & 0.65006 & \times & \times \\ \times & \times & \times & \times & 0.70641 & \times \\ \times & \times & \times & \times & \times & 0.12277 \end{bmatrix} \times 10^{-4}$$

The standardized error-estimates are calculated from

$$\begin{aligned} \frac{\hat{e}_1}{\sqrt{R'_{11}}} &= \frac{-0.02240}{\sqrt{0.48048 \times 10^{-4}}} = -3.23106 \\ \frac{\hat{e}_2}{\sqrt{R'_{22}}} &= \frac{0.00588}{\sqrt{0.81344 \times 10^{-4}}} = 0.65148 \\ \frac{\hat{e}_3}{\sqrt{R'_{33}}} &= \frac{0.22684}{\sqrt{0.22684 \times 10^{-4}}} = 0.56377 \\ \frac{\hat{e}_4}{\sqrt{R'_{44}}} &= \frac{-0.03219}{\sqrt{0.65006 \times 10^{-4}}} = -3.99261 \\ \frac{\hat{e}_5}{\sqrt{R'_{55}}} &= \frac{0.01007}{\sqrt{0.70641 \times 10^{-4}}} = 1.19820 \\ \frac{\hat{e}_6}{\sqrt{R'_{66}}} &= \frac{0.00973}{\sqrt{0.12277 \times 10^{-4}}} = 2.77670 \end{aligned}$$

from which z_4 is identified as the bad measurement. After deleting z_4 , the reduced \mathbf{H} matrix becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 0 & 12.5 & -12.5 \\ 0 & 0 & -40 \end{bmatrix}$$

The new inverse gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.17402 & 0.12540 & 0.01116 \\ 0.12540 & 0.20566 & 0.01830 \\ 0.01116 & 0.01830 & 0.05857 \end{bmatrix} \times 10^{-6}$$

where \mathbf{W} is now a 5×5 matrix which is obtained by deleting the 4th row and 4th column from the \mathbf{W} matrix specified in (a) above. The new estimates of the phase angles of the bus voltages are determined as

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.27 \\ 0.05 \end{bmatrix} = \begin{bmatrix} -0.01628 \\ -0.02286 \\ -0.00125 \end{bmatrix}$$

Conducting a Chi-square test,

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H} \hat{\mathbf{x}} = \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.27 \\ 0.05 \end{bmatrix} - \begin{bmatrix} 0.32562 \\ 0.22861 \\ 0.16450 \\ -0.27012 \\ 0.05004 \end{bmatrix} = \begin{bmatrix} -0.00562 \\ 0.01139 \\ -0.00450 \\ 0.00012 \\ -0.00004 \end{bmatrix}$$

$$\begin{aligned}
 \hat{f} &= \sum_{j=1}^5 \frac{\hat{e}_j^2}{\sigma_j^2} \\
 &= 10000 \times [(-0.00562)^2 + 0.01139^2 + (-0.00450)^2 + 0.00012^2 + (-0.00004)^2] \\
 &= 1.81584
 \end{aligned}$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi_{2,0.01}^2 = 9.21$. Since $\hat{f} < 1.81584$, we conclude that no more bad measurement exists.

15.13 In the four-bus system described in Prob. 15.12, suppose that the wattmeter on line ①-④ is out of order and that the readings of the remaining five wattmeters are the same as those specified in Prob. 15.12.

- Apply the dc power flow analysis with bus ① as reference, and determine the \mathbf{H} matrix. Then, compute the weighted least-squares estimates of the phase angles of the bus voltages in radians.
- Using the chi-square test for $\alpha = 0.01$, identify two bad measurements. Eliminate one of them and compute the weighted least-squares estimates of the bus voltage phase angles. Restore the eliminated bad measurement and remove the second one before recomputing the estimates of the bus voltage phase angles. Compare the two sets of results, and identify the buses at the estimated angles are equal in the two cases. Does the presence of line ①-④ (but with no line measurement) affect the identification of those buses? Compare the identified buses with those identified in Prob. 15.10(d).

Solution:

- Let x_1 , x_2 and x_3 denote the phase angles of the voltages at buses ②, ③ and ④, respectively. With the phase angle at bus ① equal to 0, dc power flow equations are determined to be the same as those of Prob. 15.10(a), and the corresponding \mathbf{H} matrix is given by

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.20499 & 0.18005 & 0.19554 \\ 0.18005 & 0.27980 & 0.21786 \\ 0.19554 & 0.21786 & 0.44657 \end{bmatrix} \times 10^{-6}$$

The estimates of the phase angles (in radians) of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \end{bmatrix} = \begin{bmatrix} -0.01636 \\ -0.02257 \\ 0.00073 \end{bmatrix}$$

(b) Conducting a Chi-square test,

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \end{bmatrix} - \begin{bmatrix} 0.32718 \\ 0.22565 \\ 0.15514 \\ -0.27344 \\ -0.29120 \end{bmatrix} = \begin{bmatrix} -0.00718 \\ 0.01435 \\ 0.00486 \\ -0.01656 \\ 0.02120 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^5 \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 10000 \times [(-0.00718)^2 + 0.01435^2 + 0.00486^2 + (-0.01656)^2 + 0.02120^2] \\ &= 10.0467 \end{aligned}$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi_{2,0.01}^2 = 9.21$. Since $\hat{f} > 9.21$, at least one bad measurement exists. To find the bad measurement, the diagonal elements of the matrix \mathbf{R}' are computed from

$$\begin{aligned} \mathbf{R}' &= (\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1})\mathbf{R} \\ &= \begin{bmatrix} 0.18005 & \times & \times & \times & \times \\ \times & 0.72020 & \times & \times & \times \\ \times & \times & 0.22073 & \times & \times \\ \times & \times & \times & 0.33317 & \times \\ \times & \times & \times & \times & 0.54586 \end{bmatrix} \times 10^{-4} \end{aligned}$$

The standardized error-estimates are calculated from

$$\begin{aligned} \frac{\hat{e}_1}{\sqrt{R'_{11}}} &= \frac{-0.00718}{\sqrt{0.18005 \times 10^{-4}}} = -1.69141 \\ \frac{\hat{e}_2}{\sqrt{R'_{22}}} &= \frac{0.01435}{\sqrt{0.72020 \times 10^{-4}}} = 1.69141 \\ \frac{\hat{e}_3}{\sqrt{R'_{33}}} &= \frac{0.00486}{\sqrt{0.22073 \times 10^{-4}}} = 1.03374 \\ \frac{\hat{e}_4}{\sqrt{R'_{44}}} &= \frac{-0.01656}{\sqrt{0.33317 \times 10^{-4}}} = -2.86900 \\ \frac{\hat{e}_5}{\sqrt{R'_{55}}} &= \frac{0.02120}{\sqrt{0.54586 \times 10^{-4}}} = 2.86900 \end{aligned}$$

The fourth and fifth measurements are found to be equally bad. These two measurements are discarded respectively in the following two cases.

Case A: z_4 is deleted. The reduced \mathbf{H} matrix becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The new inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.17731 \\ 0.17731 & 0.29078 & 0.29078 \\ 0.17731 & 0.29078 & 0.93078 \end{bmatrix} \times 10^{-6}$$

The new estimates of the phase angles of the bus voltages are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.27 \end{bmatrix} = \begin{bmatrix} -0.01628 \\ -0.02287 \\ -0.00127 \end{bmatrix}$$

Case B: z_5 is deleted. The reduced \mathbf{H} matrix becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \end{bmatrix}$$

The new inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.20567 \\ 0.17731 & 0.29078 & 0.17731 \\ 0.20567 & 0.17731 & 0.59630 \end{bmatrix} \times 10^{-6}$$

The new estimates of the phase angles of the bus voltages are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \end{bmatrix} = \begin{bmatrix} -0.01628 \\ -0.02287 \\ 0.00184 \end{bmatrix}$$

From the above results, the phase angles are equal for the two cases at buses ①, ② and ③, respectively. The presence of line ①-④ does not change these identified buses as long as its measurement is not accounted for in state estimation. Note that the phase angles are found to be equal at buses ①, ② and ③ in Prob. 15.10(d). Note that further calculation shows that the weighted sum-of-squares of the errors will be the same in Cases A and B above.

- 15.14 Three voltmeters and four wattmeters are installed on the three-bus system of Fig. 15.10, where per-unit reactances of the lines are $X_{12} = 0.1$, $X_{13} = 0.08$ and $X_{23} = 0.05$. The per unit values of the three voltmeter measurements are $z_1 = |V_1| = 1.01$, $z_2 = |V_2| = 1.02$ and $z_3 = |V_3| = 0.98$. The readings of the two wattmeters measuring MW generation at buses ① and ② are $z_4 = 0.48$ per unit and $z_5 = 0.33$ per unit, respectively. The measurement of the wattmeter on line ①-③ at bus ① shows $z_6 = 0.41$ per unit, and that of the wattmeter on line ②-③ at bus ② is $z_7 = 0.38$ per unit. The variances of the measurement errors are given in per unit as

$$\begin{aligned} \sigma_1^2 &= \sigma_2^2 = \sigma_3^2 = (0.02)^2 \\ \sigma_4^2 &= \sigma_5^2 = \sigma_6^2 = \sigma_7^2 = (0.05)^2 \end{aligned}$$

- (a) Use bus ① as reference to find expressions for the elements of the matrix $\mathbf{H}_x^{(k)}$ and those of the measurement errors $e_i^{(k)}$ in terms of state variables, as done in Example 15.5.

- (b) Using the initial value of $1.0 \angle 0^\circ$ per unit for all bus voltages, find the values of the state variables that will be obtained at the end of the first iteration of the weighted least-squares state estimation process.

Solution:

- (a) Five state variables are defined as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_1| \\ |V_2| \\ |V_3| \end{bmatrix}$$

The expressions of the measurement errors are

$$\begin{aligned} e_1 &= z_1 - h_1 = z_1 - |V_1| = z_1 - x_3 \\ e_2 &= z_2 - h_2 = z_2 - |V_2| = z_2 - x_4 \\ e_3 &= z_3 - h_3 = z_3 - |V_3| = z_3 - x_5 \\ e_4 &= z_4 - h_4 = z_4 - \left[\frac{|V_1||V_2|}{|Z_{12}|} \sin(\delta_1 - \delta_2) + \frac{|V_1||V_3|}{|Z_{13}|} \sin(\delta_1 - \delta_3) \right] \\ &= z_4 - \left[\frac{x_3 x_4}{0.1} \sin(0 - x_1) + \frac{x_3 x_5}{0.08} \sin(0 - x_2) \right] \\ e_5 &= z_5 - h_5 = z_5 - \left[\frac{|V_2||V_1|}{|Z_{12}|} \sin(\delta_2 - \delta_1) + \frac{|V_2||V_3|}{|Z_{23}|} \sin(\delta_2 - \delta_3) \right] \\ &= z_5 - \left[\frac{x_3 x_4}{0.1} \sin(x_1 - 0) + \frac{x_4 x_5}{0.05} \sin(x_1 - x_2) \right] \\ e_6 &= z_6 - h_6 = z_6 - \left[\frac{|V_1||V_3|}{|Z_{13}|} \sin(\delta_1 - \delta_3) \right] \\ &= z_6 - \left[\frac{x_3 x_5}{0.08} \sin(0 - x_2) \right] \\ e_7 &= z_7 - h_7 = z_7 - \left[\frac{|V_2||V_3|}{|Z_{23}|} \sin(\delta_2 - \delta_3) \right] \\ &= z_7 - \left[\frac{x_4 x_5}{0.05} \sin(x_1 - x_2) \right] \end{aligned}$$

The jacobian matrix \mathbf{H}_x is now written as

$$\mathbf{H}_x = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{\partial P_1}{\partial \delta_2} & \frac{\partial P_1}{\partial \delta_3} & \frac{\partial P_1}{\partial |V_1|} & \frac{\partial P_1}{\partial |V_2|} & \frac{\partial P_1}{\partial |V_3|} \\ \frac{\partial P_2}{\partial \delta_2} & \frac{\partial P_2}{\partial \delta_3} & \frac{\partial P_2}{\partial |V_1|} & \frac{\partial P_2}{\partial |V_2|} & \frac{\partial P_2}{\partial |V_3|} \\ \frac{\partial P_{13}}{\partial \delta_2} & \frac{\partial P_{13}}{\partial \delta_3} & \frac{\partial P_{13}}{\partial |V_1|} & \frac{\partial P_{13}}{\partial |V_2|} & \frac{\partial P_{13}}{\partial |V_3|} \\ \frac{\partial P_{23}}{\partial \delta_2} & \frac{\partial P_{23}}{\partial \delta_3} & \frac{\partial P_{23}}{\partial |V_1|} & \frac{\partial P_{23}}{\partial |V_2|} & \frac{\partial P_{23}}{\partial |V_3|} \end{bmatrix}$$

$$H_X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{-|V_1||V_2|\cos(\delta_1-\delta_2)}{|Z_{12}|} & \frac{-|V_1||V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & \left(\frac{|V_2|\sin(\delta_1-\delta_2)}{|Z_{12}|} + \frac{|V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|} \right) & \frac{|V_1|\sin(\delta_1-\delta_2)}{|Z_{12}|} & \frac{|V_1|\sin(\delta_1-\delta_3)}{|Z_{13}|} \\ \left(\frac{|V_2||V_1|\cos(\delta_2-\delta_1)}{|Z_{12}|} + \frac{|V_3||V_1|\cos(\delta_2-\delta_3)}{|Z_{13}|} \right) & \frac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{|V_2|\sin(\delta_2-\delta_1)}{|Z_{12}|} & \left(\frac{|V_1|\sin(\delta_2-\delta_1)}{|Z_{12}|} + \frac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} \right) & \frac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|} \\ 0 & \frac{-|V_1||V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & \frac{|V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|} & 0 & \frac{|V_1|\sin(\delta_1-\delta_3)}{|Z_{13}|} \\ \frac{|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \frac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \frac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -10x_3x_4\cos(-x_1) & -12.5x_3x_5\cos(-x_2) & \left(\frac{10x_4\sin(-x_1)}{+12.5x_5\sin(-x_2)} \right) & 10x_3\sin(-x_1) & 12.5x_3\sin(-x_2) \\ \left(\frac{10x_3x_4\cos(x_1)}{+20x_4x_5\cos(x_1-x_2)} \right) & -20x_4x_5\cos(x_1-x_2) & 10x_4\sin(x_1) & \left(\frac{10x_4\sin(x_1)}{+20x_5\sin(x_1-x_2)} \right) & 20x_4\sin(x_1-x_2) \\ 0 & -12.5x_3x_5\cos(-x_2) & 12.5x_5\sin(-x_2) & 0 & 12.5x_3\sin(-x_2) \\ 20x_4x_5\cos(x_1-x_2) & -20x_4x_5\cos(x_1-x_2) & 0 & 20x_5\sin(x_1-x_2) & 20x_4\sin(x_1-x_2) \end{bmatrix}$$

(b) Using flat-start values,

$$H_X^{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -10 & -12.5 & 0 & 0 & 0 \\ 30 & -20 & 0 & 0 & 0 \\ 0 & -12.5 & 0 & 0 & 0 \\ 20 & -20 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} e_1^{(0)} &= z_1 - h_1^{(0)} = 1.01 - 1.0 = 0.01 \\ e_2^{(0)} &= z_2 - h_2^{(0)} = 1.02 - 1.0 = 0.02 \\ e_3^{(0)} &= z_3 - h_3^{(0)} = 0.98 - 1.0 = -0.02 \\ e_4^{(0)} &= z_4 - h_4^{(0)} = 0.48 - 0 = 0.48 \\ e_5^{(0)} &= z_5 - h_5^{(0)} = 0.33 - 0 = 0.33 \\ e_6^{(0)} &= z_6 - h_6^{(0)} = 0.41 - 0 = 0.41 \end{aligned}$$

$$e_7^{(0)} = z_7 - h_7^{(0)} = 0.38 - 0 = 0.38$$

Note that

$$\mathbf{R}^{-1} = \begin{bmatrix} \frac{1}{0.02^2} & \times & \times & \times & \times & \times & \times \\ \times & \frac{1}{0.02^2} & \times & \times & \times & \times & \times \\ \times & \times & \frac{1}{0.02^2} & \times & \times & \times & \times \\ \times & \times & \times & \frac{1}{0.05^2} & \times & \times & \times \\ \times & \times & \times & \times & \frac{1}{0.05^2} & \times & \times \\ \times & \times & \times & \times & \times & \frac{1}{0.05^2} & \times \\ \times & \times & \times & \times & \times & \times & \frac{1}{0.05^2} \end{bmatrix}$$

Using \mathbf{R}^{-1} and $\mathbf{H}_x^{(0)}$ obtained above,

$$\left[\mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \mathbf{H}_x^{(0)} \right]^{-1} = \begin{bmatrix} 0.3512 \times 10^{-5} & 0.2762 \times 10^{-5} & -0.3843 \times 10^{-10} & 0.3297 \times 10^{-10} & 0.3553 \times 10^{-18} \\ 0.2762 \times 10^{-5} & 0.4420 \times 10^{-5} & -0.5400 \times 10^{-10} & -0.2318 \times 10^{-10} & -0.1748 \times 10^{-10} \\ -0.3843 \times 10^{-10} & -0.5400 \times 10^{-10} & 0.4000 \times 10^{-3} & -0.3382 \times 10^{-16} & 0.5217 \times 10^{-23} \\ 0.3297 \times 10^{-10} & -0.2318 \times 10^{-10} & -0.3382 \times 10^{-16} & 0.4000 \times 10^{-3} & -0.2413 \times 10^{-22} \\ 0.3553 \times 10^{-18} & -0.1748 \times 10^{-10} & -0.5217 \times 10^{-23} & -0.2413 \times 10^{-22} & 0.4000 \times 10^{-3} \end{bmatrix}$$

Finally, we have

$$\begin{aligned} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \\ x_5^{(1)} \end{bmatrix} &= \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \\ x_5^{(0)} \end{bmatrix} + \left(\mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \mathbf{H}_x^{(0)} \right)^{-1} \mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \begin{bmatrix} e_1^{(0)} \\ e_2^{(0)} \\ e_3^{(0)} \\ e_4^{(0)} \\ e_5^{(0)} \\ e_6^{(0)} \\ e_7^{(0)} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.01014 \\ -0.03074 \\ 0.01000 \\ 0.02000 \\ -0.02000 \end{bmatrix} = \begin{bmatrix} -0.01014 \text{ radian} \\ -0.03074 \text{ radian} \\ 1.01 \text{ per unit} \\ 1.02 \text{ per unit} \\ 0.98 \text{ per unit} \end{bmatrix} \end{aligned}$$

- 15.15 Application of the weighted least-squares state estimation to the three-bus system with all the measurements described in Prob. 15.14 yields the following estimates of the states

$$\begin{aligned} |V_1| &= 1.0109 \text{ per unit} \\ |V_2| &= 1.0187 \text{ per unit} & \delta_2 &= -0.0101 \text{ radians} \\ |V_3| &= 0.9804 \text{ per unit} & \delta_3 &= -0.0308 \text{ radians} \end{aligned}$$

The diagonal elements in the covariance matrix \mathbf{R}' are 0.8637×10^{-6} , 0.1882×10^{-5} , 0.2189×10^{-6} , 0.7591×10^{-3} , 0.8786×10^{-3} , 0.1812×10^{-2} and 0.1532×10^{-2} . Find the estimates of the measurement errors \hat{e}_i , and the corresponding standardized errors.

Solution:

Refer to the solution of Prob. 15.14 for expressions for $h(x)$. The estimates of the measurement errors are

$$\begin{aligned}
 \hat{e}_1 &= z_1 - h_1(x) = z_1 - |V_1| = 1.01 - 1.0109 = -0.0009 \\
 \hat{e}_2 &= z_2 - h_2(x) = z_2 - |V_2| = 1.02 - 1.0187 = 0.0013 \\
 \hat{e}_3 &= z_3 - h_3(x) = z_3 - |V_3| = 0.98 - 0.9804 = -0.0004 \\
 \hat{e}_4 &= z_4 - h_4(x) = 0.48 - [10 \times 1.0109 \times 1.0187 \times \sin(0.0101) \\
 &\quad + 12.5 \times 1.0109 \times 0.9804 \times \sin(0.0308)] \\
 &= 0.48 - 0.1040 - 0.3815 = -0.0055 \\
 \hat{e}_5 &= z_5 - h_5(x) = 0.33 - [10 \times 1.0187 \times 1.0109 \times \sin(-0.0101) \\
 &\quad + 20 \times 1.0187 \times 0.9804 \times \sin(-0.0101 + 0.0308)] \\
 &= 0.33 + 0.1040 - 0.4134 = 0.0206 \\
 \hat{e}_6 &= z_6 - h_6(x) = 0.41 - [12.5 \times 1.0109 \times 0.9804 \times \sin(0.0308)] \\
 &= 0.41 - 0.3815 = 0.0285 \\
 \hat{e}_7 &= z_7 - h_7(x) = 0.38 - [20 \times 1.0187 \times 0.9804 \times \sin(-0.0101 + 0.0308)] \\
 &= 0.38 - 0.4134 = -0.0334
 \end{aligned}$$

The standardized errors become

$$\begin{aligned}
 \frac{\hat{e}_1}{\sqrt{R'_{11}}} &= \frac{-0.0009}{\sqrt{0.8637 \times 10^{-6}}} = -0.9684 \\
 \frac{\hat{e}_2}{\sqrt{R'_{22}}} &= \frac{0.0013}{\sqrt{0.1882 \times 10^{-5}}} = 0.9476 \\
 \frac{\hat{e}_3}{\sqrt{R'_{33}}} &= \frac{-0.0004}{\sqrt{0.2189 \times 10^{-6}}} = -0.8549 \\
 \frac{\hat{e}_4}{\sqrt{R'_{44}}} &= \frac{-0.0055}{\sqrt{0.7591 \times 10^{-3}}} = -0.1996 \\
 \frac{\hat{e}_5}{\sqrt{R'_{55}}} &= \frac{0.0206}{\sqrt{0.8786 \times 10^{-3}}} = 0.6950 \\
 \frac{\hat{e}_6}{\sqrt{R'_{66}}} &= \frac{0.0285}{\sqrt{0.1812 \times 10^{-2}}} = 0.6695 \\
 \frac{\hat{e}_7}{\sqrt{R'_{77}}} &= \frac{-0.0334}{\sqrt{0.1532 \times 10^{-2}}} = -0.8533
 \end{aligned}$$

- 15.16 Solve Prob. 15.14 when the two wattmeters installed on lines ①-③ and ②-③ are replaced with two varmeters and their readings are 0.08 and 0.24 per unit, respectively.

Solution:

(a) Five state variables are defined as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_1| \\ |V_2| \\ |V_3| \end{bmatrix}$$

The expressions of the measurement errors are

$$\begin{aligned} e_1 &= z_1 - h_1 = z_1 - |V_1| = z_1 - x_3 \\ e_2 &= z_2 - h_2 = z_2 - |V_2| = z_2 - x_4 \\ e_3 &= z_3 - h_3 = z_3 - |V_3| = z_3 - x_5 \\ e_4 &= z_4 - h_4 = z_4 - \left[\frac{|V_1||V_2|}{|Z_{12}|} \sin(\delta_1 - \delta_2) + \frac{|V_1||V_3|}{|Z_{13}|} \sin(\delta_1 - \delta_3) \right] \\ &= z_4 - \left[\frac{x_3 x_4}{0.1} \sin(0 - x_1) + \frac{x_3 x_5}{0.08} \sin(0 - x_2) \right] \\ e_5 &= z_5 - h_5 = z_5 - \left[\frac{|V_2||V_1|}{|Z_{12}|} \sin(\delta_2 - \delta_1) + \frac{|V_2||V_3|}{|Z_{23}|} \sin(\delta_2 - \delta_3) \right] \\ &= z_5 - \left[\frac{x_3 x_4}{0.1} \sin(x_1 - 0) + \frac{x_4 x_5}{0.05} \sin(x_1 - x_2) \right] \\ e_6 &= z_6 - h_6 = z_6 - \left[\frac{|V_1|^2}{|Z_{13}|} - \frac{|V_1||V_3|}{|Z_{13}|} \cos(\delta_1 - \delta_3) \right] \\ &= z_6 - \left[\frac{(x_3)^2}{0.08} - \frac{x_3 x_5}{0.08} \cos(0 - x_2) \right] \\ e_7 &= z_7 - h_7 = z_7 - \left[\frac{|V_2|^2}{|Z_{23}|} - \frac{|V_2||V_3|}{|Z_{23}|} \cos(\delta_2 - \delta_3) \right] \\ &= z_7 - \left[\frac{(x_4)^2}{0.05} - \frac{x_4 x_5}{0.05} \cos(x_1 - x_2) \right] \end{aligned}$$

The jacobian matrix \mathbf{H}_x is now written as

$$\mathbf{H}_x = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{\partial P_1}{\partial \delta_2} & \frac{\partial P_1}{\partial \delta_3} & \frac{\partial P_1}{\partial |V_1|} & \frac{\partial P_1}{\partial |V_2|} & \frac{\partial P_1}{\partial |V_3|} \\ \frac{\partial P_2}{\partial \delta_2} & \frac{\partial P_2}{\partial \delta_3} & \frac{\partial P_2}{\partial |V_1|} & \frac{\partial P_2}{\partial |V_2|} & \frac{\partial P_2}{\partial |V_3|} \\ \frac{\partial Q_{13}}{\partial \delta_2} & \frac{\partial Q_{13}}{\partial \delta_3} & \frac{\partial Q_{13}}{\partial |V_1|} & \frac{\partial Q_{13}}{\partial |V_2|} & \frac{\partial Q_{13}}{\partial |V_3|} \\ \frac{\partial Q_{23}}{\partial \delta_2} & \frac{\partial Q_{23}}{\partial \delta_3} & \frac{\partial Q_{23}}{\partial |V_1|} & \frac{\partial Q_{23}}{\partial |V_2|} & \frac{\partial Q_{23}}{\partial |V_3|} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{-|V_1||V_2|\cos(\delta_1-\delta_2)}{|Z_{12}|} & \frac{-|V_1||V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & \left(\frac{|V_2|\sin(\delta_1-\delta_2)}{|Z_{12}|} + \frac{|V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|} \right) & \frac{|V_1|\sin(\delta_1-\delta_2)}{|Z_{12}|} & \frac{|V_1|\sin(\delta_1-\delta_3)}{|Z_{13}|} \\ \left(\frac{|V_2||V_1|\cos(\delta_2-\delta_1)}{|Z_{12}|} + \frac{|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} \right) & \frac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{|V_2|\sin(\delta_2-\delta_1)}{|Z_{12}|} & \left(\frac{|V_1|\sin(\delta_2-\delta_1)}{|Z_{12}|} + \frac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} \right) & \frac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|} \\ 0 & \frac{-|V_1||V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|} & \frac{2|V_1|}{Z_{13}} - \frac{|V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & 0 & \frac{-|V_1|\cos(\delta_1-\delta_3)}{|Z_{13}|} \\ \frac{|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \frac{2|V_2|}{Z_{23}} - \frac{|V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2|\cos(\delta_2-\delta_3)}{|Z_{23}|} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -10x_3x_4\cos(-x_1) & -12.5x_3x_5\cos(-x_2) & \left(\frac{10x_4\sin(-x_1)}{+12.5x_5\sin(-x_2)} \right) & 10x_3\sin(-x_1) & 12.5x_3\sin(-x_2) \\ \left(\frac{10x_3x_4\cos(x_1)}{+20x_4x_5\cos(x_1-x_2)} \right) & -20x_4x_5\cos(x_1-x_2) & 10x_4\sin(x_1) & \left(\frac{10x_4\sin(x_1)}{+20x_5\sin(x_1-x_2)} \right) & 20x_4\sin(x_1-x_2) \\ 0 & -12.5x_3x_5\sin(-x_2) & 25x_3-12.5x_5\cos(-x_2) & 0 & -12.5x_3\cos(-x_2) \\ 20x_4x_5\sin(x_1-x_2) & -20x_4x_5\sin(x_1-x_2) & 0 & 40x_4-20x_5\cos(x_1-x_2) & -20x_4\cos(x_1-x_2) \end{bmatrix}$$

(b) Using flat-start values,

$$\mathbf{H}_X^{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -10 & -12.5 & 0 & 0 & 0 \\ 30 & -20 & 0 & 0 & 0 \\ 0 & 0 & 12.5 & 0 & -12.5 \\ 0 & 0 & 0 & 20 & -20 \end{bmatrix}$$

$$\begin{aligned} e_1^{(0)} &= z_1 - h_1^{(0)} = 1.01 - 1.0 = 0.01 \\ e_2^{(0)} &= z_2 - h_2^{(0)} = 1.02 - 1.0 = 0.02 \\ e_3^{(0)} &= z_3 - h_3^{(0)} = 0.98 - 1.0 = -0.02 \\ e_4^{(0)} &= z_4 - h_4^{(0)} = 0.48 - 0 = 0.48 \\ e_5^{(0)} &= z_5 - h_5^{(0)} = 0.33 - 0 = 0.33 \\ e_6^{(0)} &= z_6 - h_6^{(0)} = 0.08 - 0 = 0.08 \end{aligned}$$

$$e_7^{(0)} = z_7 - h_7^{(0)} = 0.24 - 0 = 0.24$$

Note that

$$\mathbf{R}^{-1} = \begin{bmatrix} \frac{1}{0.02^2} & \times & \times & \times & \times & \times & \times \\ \times & \frac{1}{0.02^2} & \times & \times & \times & \times & \times \\ \times & \times & \frac{1}{0.02^2} & \times & \times & \times & \times \\ \times & \times & \times & \frac{1}{0.05^2} & \times & \times & \times \\ \times & \times & \times & \times & \frac{1}{0.05^2} & \times & \times \\ \times & \times & \times & \times & \times & \frac{1}{0.05^2} & \times \\ \times & \times & \times & \times & \times & \times & \frac{1}{0.05^2} \end{bmatrix}$$

Using \mathbf{R}^{-1} and $\mathbf{H}_x^{(0)}$ obtained above,

$$\left[\mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \mathbf{H}_x^{(0)} \right]^{-1} = \begin{bmatrix} 0.4206 \times 10^{-5} & 0.3592 \times 10^{-5} & -0.4194 \times 10^{-11} & -0.2214 \times 10^{-11} & -0.2715 \times 10^{-11} \\ 0.3592 \times 10^{-5} & 0.7561 \times 10^{-5} & -0.4035 \times 10^{-10} & -0.3918 \times 10^{-10} & -0.3905 \times 10^{-10} \\ -0.4194 \times 10^{-11} & -0.4035 \times 10^{-10} & 0.1409 \times 10^{-3} & 0.1285 \times 10^{-3} & 0.1305 \times 10^{-3} \\ -0.2214 \times 10^{-11} & -0.3918 \times 10^{-10} & 0.1285 \times 10^{-3} & 0.1378 \times 10^{-3} & 0.1337 \times 10^{-3} \\ -0.2715 \times 10^{-11} & -0.3905 \times 10^{-10} & 0.1305 \times 10^{-3} & 0.1337 \times 10^{-3} & 0.1358 \times 10^{-3} \end{bmatrix}$$

Finally, we have

$$\begin{aligned} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \\ x_5^{(1)} \end{bmatrix} &= \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \\ x_5^{(0)} \end{bmatrix} + \left(\mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \mathbf{H}_x^{(0)} \right)^{-1} \mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \begin{bmatrix} e_1^{(0)} \\ e_2^{(0)} \\ e_3^{(0)} \\ e_4^{(0)} \\ e_5^{(0)} \\ e_6^{(0)} \\ e_7^{(0)} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.00952 \\ -0.03078 \\ 0.00371 \\ 0.00923 \\ -0.00294 \end{bmatrix} = \begin{bmatrix} -0.00952 \text{ radian} \\ -0.03078 \text{ radian} \\ 1.00371 \text{ per unit} \\ 1.00923 \text{ per unit} \\ 0.99706 \text{ per unit} \end{bmatrix} \end{aligned}$$

15.17 Suppose that real and reactive power flows are measured at both ends of each of the five lines in the four-bus system of Fig. 15.9 using ten wattmeters and ten varmeters. The voltage magnitude is measured at bus ② only, and bus injected powers are not measured at all.

- Determine the structure of \mathbf{H}_x by writing the partial derivative form a its non-zero elements, as shown in Example 15.8. Assume that line flow measurements are ordered in the following sequence: ①-②, ①-③, ②-③, ②-④ and ③-④ (and the same sequence also in reverse directions).
- Suppose that the elements of the \mathbf{Y}_{bus} of the network is given by

$$Y_{ij} = G_{ij} + jB_{ij} = |Y_{ij}| \angle \theta_{ij}$$

and that the total charging susceptance of line ①-③ is B'_{ij} . Write out nonlinear functions which express the measured quantities P_{21} and Q_{21} in terms of state variables.

- (c) Write out the expressions, similar to those given in Example 15.8, for the non-zero elements in the rows of the matrix \mathbf{H}_x corresponding to measurements P_{21} and Q_{21} in terms of state variables.

Solution:

(a)

$$\mathbf{H}_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{\partial P_{12}}{\partial \delta_2} & 0 & 0 & \frac{\partial P_{12}}{\partial |V_1|} & \frac{\partial P_{12}}{\partial |V_2|} & 0 & 0 \\ 0 & \frac{\partial P_{13}}{\partial \delta_3} & 0 & \frac{\partial P_{13}}{\partial |V_1|} & 0 & \frac{\partial P_{13}}{\partial |V_3|} & 0 \\ \frac{\partial P_{23}}{\partial \delta_2} & \frac{\partial P_{23}}{\partial \delta_3} & 0 & 0 & \frac{\partial P_{23}}{\partial |V_2|} & \frac{\partial P_{23}}{\partial |V_3|} & 0 \\ \frac{\partial P_{24}}{\partial \delta_2} & 0 & \frac{\partial P_{24}}{\partial \delta_4} & 0 & \frac{\partial P_{24}}{\partial |V_2|} & 0 & \frac{\partial P_{24}}{\partial |V_4|} \\ 0 & \frac{\partial P_{34}}{\partial \delta_3} & \frac{\partial P_{34}}{\partial \delta_4} & 0 & 0 & \frac{\partial P_{34}}{\partial |V_3|} & \frac{\partial P_{34}}{\partial |V_4|} \\ \frac{\partial P_{21}}{\partial \delta_2} & 0 & 0 & \frac{\partial P_{21}}{\partial |V_1|} & \frac{\partial P_{21}}{\partial |V_2|} & 0 & 0 \\ 0 & \frac{\partial P_{31}}{\partial \delta_3} & 0 & \frac{\partial P_{31}}{\partial |V_1|} & 0 & \frac{\partial P_{31}}{\partial |V_3|} & 0 \\ \frac{\partial P_{32}}{\partial \delta_2} & \frac{\partial P_{32}}{\partial \delta_3} & 0 & 0 & \frac{\partial P_{32}}{\partial |V_2|} & \frac{\partial P_{32}}{\partial |V_3|} & 0 \\ \frac{\partial P_{42}}{\partial \delta_2} & 0 & \frac{\partial P_{42}}{\partial \delta_4} & 0 & \frac{\partial P_{42}}{\partial |V_2|} & 0 & \frac{\partial P_{42}}{\partial |V_4|} \\ 0 & \frac{\partial P_{43}}{\partial \delta_3} & \frac{\partial P_{43}}{\partial \delta_4} & 0 & 0 & \frac{\partial P_{43}}{\partial |V_3|} & \frac{\partial P_{43}}{\partial |V_4|} \\ \frac{\partial Q_{12}}{\partial \delta_2} & 0 & 0 & \frac{\partial Q_{12}}{\partial |V_1|} & \frac{\partial Q_{12}}{\partial |V_2|} & 0 & 0 \\ 0 & \frac{\partial Q_{13}}{\partial \delta_3} & 0 & \frac{\partial Q_{13}}{\partial |V_1|} & 0 & \frac{\partial Q_{13}}{\partial |V_3|} & 0 \\ \frac{\partial Q_{23}}{\partial \delta_2} & \frac{\partial Q_{23}}{\partial \delta_3} & 0 & 0 & \frac{\partial Q_{23}}{\partial |V_2|} & \frac{\partial Q_{23}}{\partial |V_3|} & 0 \\ \frac{\partial Q_{24}}{\partial \delta_2} & 0 & \frac{\partial Q_{24}}{\partial \delta_4} & 0 & \frac{\partial Q_{24}}{\partial |V_2|} & 0 & \frac{\partial Q_{24}}{\partial |V_4|} \\ 0 & \frac{\partial Q_{34}}{\partial \delta_3} & \frac{\partial Q_{34}}{\partial \delta_4} & 0 & 0 & \frac{\partial Q_{34}}{\partial |V_3|} & \frac{\partial Q_{34}}{\partial |V_4|} \\ \frac{\partial Q_{21}}{\partial \delta_2} & 0 & 0 & \frac{\partial Q_{21}}{\partial |V_1|} & \frac{\partial Q_{21}}{\partial |V_2|} & 0 & 0 \\ 0 & \frac{\partial Q_{31}}{\partial \delta_3} & 0 & \frac{\partial Q_{31}}{\partial |V_1|} & 0 & \frac{\partial Q_{31}}{\partial |V_3|} & 0 \\ \frac{\partial Q_{32}}{\partial \delta_2} & \frac{\partial Q_{32}}{\partial \delta_3} & 0 & 0 & \frac{\partial Q_{32}}{\partial |V_2|} & \frac{\partial Q_{32}}{\partial |V_3|} & 0 \\ \frac{\partial Q_{42}}{\partial \delta_2} & 0 & \frac{\partial Q_{42}}{\partial \delta_4} & 0 & \frac{\partial Q_{42}}{\partial |V_2|} & 0 & \frac{\partial Q_{42}}{\partial |V_4|} \\ 0 & \frac{\partial Q_{43}}{\partial \delta_3} & \frac{\partial Q_{43}}{\partial \delta_4} & 0 & 0 & \frac{\partial Q_{43}}{\partial |V_3|} & \frac{\partial Q_{43}}{\partial |V_4|} \end{bmatrix}$$

(b)

$$\begin{aligned} P_{21} &= -|V_2|^2 G_{21} + |V_2||V_1||Y_{21}|\cos(\theta_{21} + \delta_1 - \delta_2) \\ Q_{21} &= |V_2|^2 B_{21} - |V_2|^2 \left(\frac{B'_{21}}{2} \right) - |V_2||V_1||Y_{21}|\sin(\theta_{21} + \delta_1 - \delta_2) \end{aligned}$$

(c)

$$\begin{aligned}
\frac{\partial P_{21}}{\partial \delta_2} &= |V_2||V_1||Y_{21}|\sin(\theta_{21} - \delta_2) \\
\frac{\partial P_{21}}{\partial |V_1|} &= |V_2||Y_{21}|\cos(\theta_{21} - \delta_2) \\
\frac{\partial P_{21}}{\partial |V_2|} &= -2|V_2|G_{21} + |V_1||Y_{21}|\cos(\theta_{21} - \delta_2) \\
\frac{\partial Q_{21}}{\partial \delta_2} &= |V_2||V_1||Y_{21}|\cos(\theta_{21} - \delta_2) \\
\frac{\partial Q_{21}}{\partial |V_1|} &= -|V_2||Y_{21}|\sin(\theta_{21} - \delta_2) \\
\frac{\partial Q_{21}}{\partial |V_2|} &= 2|V_2|B_{21} - |V_2|B'_{21} - |V_1||Y_{21}|\sin(\theta_{21} - \delta_2)
\end{aligned}$$

where $\delta_1 \triangleq 0$

15.18 The method of Example 15.8 based on measurements of only line flows (plus a voltage measurement at one bus) is applied to the three-bus system of Fig. 15.10 using three wattmeters and three varmeters. The per unit values of the measurements are

$$\begin{aligned}
z_1 = |V_1| &= 1.0 & z_5 = Q_{12} &= -0.101 \\
z_2 = P_{12} &= 0.097 & z_6 = Q_{13} &= 0.048 \\
z_3 = P_{13} &= 0.383 & z_7 = Q_{23} &= 0.276 \\
z_4 = P_{23} &= 0.427
\end{aligned}$$

where the variances of all the measurements are 0.02^2 . The per-unit reactances of the lines are as specified in Prob. 15.14. Using bus ① as reference and flat-start values, find the values of the state variables that will be obtained at the end of the first iteration of the weighted least-squares state estimation.

Solution:

The measurement errors are

$$\begin{aligned}
e_1 = z_1 - h_1 &= z_1 - |V_1| \\
e_2 = z_2 - h_2 &= z_2 - P_{12} = z_2 - \frac{|V_1||V_2|}{|Z_{12}|}\sin(\delta_1 - \delta_2) \\
e_3 = z_3 - h_3 &= z_3 - P_{13} = z_3 - \frac{|V_1||V_3|}{|Z_{13}|}\sin(\delta_1 - \delta_3) \\
e_4 = z_4 - h_4 &= z_4 - P_{23} = z_4 - \frac{|V_2||V_3|}{|Z_{23}|}\sin(\delta_2 - \delta_3) \\
e_5 = z_5 - h_5 &= z_5 - Q_{12} = z_5 - \left[\frac{|V_1|^2}{|Z_{12}|} - \frac{|V_1||V_2|}{|Z_{12}|}\cos(\delta_1 - \delta_2) \right] \\
e_6 = z_6 - h_6 &= z_6 - Q_{13} = z_6 - \left[\frac{|V_1|^2}{|Z_{13}|} - \frac{|V_1||V_3|}{|Z_{13}|}\cos(\delta_1 - \delta_3) \right] \\
e_7 = z_7 - h_7 &= z_7 - Q_{23} = z_7 - \left[\frac{|V_2|^2}{|Z_{23}|} - \frac{|V_2||V_3|}{|Z_{23}|}\cos(\delta_2 - \delta_3) \right]
\end{aligned}$$

Using the flat-start values, the initial measurement errors become

$$\begin{aligned} e_1^{(0)} &= 1.0 - 1.0 = 0 \\ e_2^{(0)} &= 0.097 - 0 = 0.097 \\ e_3^{(0)} &= 0.383 - 0 = 0.383 \\ e_4^{(0)} &= 0.427 - 0 = 0.427 \\ e_5^{(0)} &= -0.101 - \left[\frac{1}{0.1} - \frac{1}{0.1} \right] = -0.101 \\ e_6^{(0)} &= 0.048 - \left[\frac{1}{0.08} - \frac{1}{0.08} \right] = 0.048 \\ e_7^{(0)} &= 0.276 - \left[\frac{1}{0.05} - \frac{1}{0.05} \right] = 0.276 \end{aligned}$$

The jacobian matrix $\mathbf{H}_{\mathbf{x}}$ is given by

$$\mathbf{H}_X = \begin{bmatrix} \delta_2 & \delta_3 & |V_1| & |V_2| & |V_3| \\ 0 & 0 & 1 & 0 & 0 \\ \frac{-|V_1||V_2|\cos(-\delta_2)}{|Z_{12}|} & 0 & \frac{|V_2|\sin(-\delta_2)}{|Z_{12}|} & \frac{|V_1|\sin(-\delta_2)}{|Z_{12}|} & 0 \\ 0 & \frac{-|V_1||V_3|\cos(-\delta_3)}{|Z_{13}|} & \frac{|V_3|\sin(-\delta_3)}{|Z_{13}|} & 0 & \frac{|V_1|\sin(-\delta_3)}{|Z_{13}|} \\ \frac{|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \frac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \frac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|} \\ \frac{-|V_1||V_2|\sin(-\delta_2)}{|Z_{12}|} & 0 & \left(\frac{2|V_1|}{|Z_{12}|} - \frac{|V_2|\cos(-\delta_2)}{|Z_{12}|} \right) & \frac{-|V_1|\cos(-\delta_2)}{|Z_{12}|} & 0 \\ 0 & \frac{-|V_1||V_3|\sin(-\delta_3)}{|Z_{13}|} & \left(\frac{2|V_1|}{|Z_{13}|} - \frac{|V_3|\cos(-\delta_3)}{|Z_{13}|} \right) & 0 & \frac{-|V_1|\cos(-\delta_3)}{|Z_{13}|} \\ \frac{|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \left(\frac{2|V_2|}{|Z_{23}|} - \frac{|V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} \right) & \frac{-|V_2|\cos(\delta_2-\delta_3)}{|Z_{23}|} \end{bmatrix}$$

where $\delta_1 \triangleq 0$.

$$\mathbf{H}_x^{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 \\ 0 & -12.5 & 0 & 0 & 0 \\ 20 & -20 & 0 & 0 & 0 \\ 0 & 0 & 10 & -10 & 0 \\ 0 & 0 & 12.5 & 0 & -12.5 \\ 0 & 0 & 0 & 20 & -20 \end{bmatrix}$$

Note that

[illegible]

Using $\mathbf{H}_x^{(0)}$ obtained above, we have

$$\left[\mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \mathbf{H}_x^{(0)} \right]^{-1} = \begin{bmatrix} 0.1884 \times 10^{-5} & 0.1355 \times 10^{-5} & 0.5578 \times 10^{-18} & 0.5571 \times 10^{-18} & 0.5583 \times 10^{-18} \\ & 0.1693 \times 10^{-5} & 0.4179 \times 10^{-18} & 0.4184 \times 10^{-18} & 0.4176 \times 10^{-18} \\ & & 0.4000 \times 10^{-3} & 0.4000 \times 10^{-3} & 0.4000 \times 10^{-3} \\ & & & 0.4019 \times 10^{-3} & 0.4014 \times 10^{-3} \\ & & & & 0.4017 \times 10^{-3} \end{bmatrix}$$

Finally, we have

$$\begin{bmatrix} \delta_2^{(1)} \\ \delta_3^{(1)} \\ |V_1|^{(1)} \\ |V_2|^{(1)} \\ |V_3|^{(1)} \end{bmatrix} = \begin{bmatrix} \delta_2^{(0)} \\ \delta_3^{(0)} \\ |V_1|^{(0)} \\ |V_2|^{(0)} \\ |V_3|^{(0)} \end{bmatrix} + \left(\mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \mathbf{H}_x^{(0)} \right)^{-1} \mathbf{H}_x^{(0)T} \mathbf{R}^{-1} \begin{bmatrix} e_1^{(0)} \\ e_2^{(0)} \\ e_3^{(0)} \\ e_4^{(0)} \\ e_5^{(0)} \\ e_6^{(0)} \\ e_7^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.00948 \\ -0.03078 \\ 0.0 \\ 0.01003 \\ -0.00379 \end{bmatrix} = \begin{bmatrix} -0.00948 \text{ radian} \\ -0.03078 \text{ radian} \\ 1.0 \text{ per unit} \\ 1.01003 \text{ per unit} \\ 0.99621 \text{ per unit} \end{bmatrix}$$

Chapter 16 Problem Solutions

- 16.1 A 60-Hz four-pole turbogenerator rated 500 MVA, 22 kV has an inertia constant of $H = 7.5$ MJ/MVA. Find (a) the kinetic energy stored in the rotor at synchronous speed and (b) the angular acceleration if the electrical power developed is 400 MW when the input less the rotational losses is 740,000 hp.

Solution:

(a) Kinetic energy = $500 \times 7.5 = 3750$ MJ

(b) Input power = $740,000 \times 746 \times 10^{-6} = 552$ MW. By Eq. (16.14),

$$\begin{aligned} \text{Input power} - \text{rotational loss} &= \frac{7.5}{180 \times 60} \frac{d^2 \delta}{dt^2} = \frac{552 - 400}{500} \\ \frac{d^2 \delta}{dt^2} &= 437.8 \text{ elec. degrees/s}^2 \end{aligned}$$

For a four-pole machine,

$$\begin{aligned} \frac{d^2 \delta}{dt^2} &= \frac{437.8}{2} = 218.9 \text{ mech. degrees/s}^2 \\ \text{or } 60 \times \frac{218.9}{360} &= 36.5 \text{ rpm/s}^2 \end{aligned}$$