Chapter 8 Problem Solutions

8.1 Form $\mathbf{Z}_{\text{bus}}$ for the circuit of Fig. 8.13 after removing node ⑤ by converting the voltage source to a current source. Determine the voltages with respect to reference node at each of the four other nodes when $V = 1.2\angle 0°$ and the load currents are $I_{L1} = -j0.1$, $I_{L2} = -j0.1$, $I_{L3} = -j0.2$ and $I_{L4} = -j0.2$, all in per unit.

Solution:

When the voltage source is converted to a current source and added as an injected current at that node, the voltages are

$$\mathbf{Z}_{\text{bus}}\mathbf{I} = \begin{bmatrix} j0.2 & j0.2 & j0.2 & j0.2 \\ j0.2 & j0.6 & j0.2 & j0.6 \\ j0.2 & j0.2 & j0.8 & j0.2 \\ j0.2 & j0.6 & j0.2 & j1.1 \end{bmatrix} \begin{bmatrix} -j5.9 \\ j0.1 \\ j0.2 \\ j0.2 \end{bmatrix} = \begin{bmatrix} 1.08 \\ 0.96 \\ 0.96 \\ 0.86 \end{bmatrix} \text{ per unit}$$

Alternatively, if the high-side of the voltage source is chosen as the reference:

$$\mathbf{Z}_{\text{bus}}\mathbf{I} = \begin{bmatrix} j0.2 & j0.2 & j0.2 & j0.2 \\ j0.2 & j0.6 & j0.2 & j0.6 \\ j0.2 & j0.2 & j0.8 & j0.2 \\ j0.2 & j0.6 & j0.2 & j1.1 \end{bmatrix} \begin{bmatrix} j0.1 \\ j0.1 \\ j0.2 \\ j0.2 \end{bmatrix} = \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix} \text{ per unit}$$

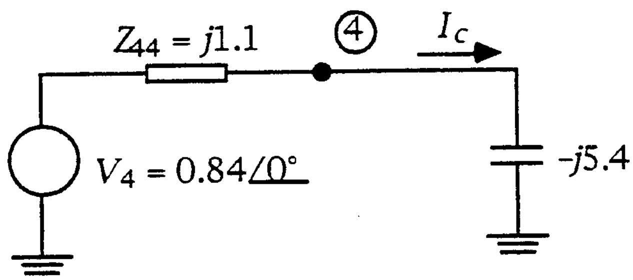
and the bus voltages are

$$1.2 \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix} = \begin{bmatrix} 1.08 \\ 0.96 \\ 0.96 \\ 0.86 \end{bmatrix} \text{ per unit}$$

(8.2) From the solution of Prob. 8.1, draw the Thévénin equivalent circuit at bus 4 of Fig. 8.13 and use it to determine the current drawn by a capacitor of reactance 5.4 per-unit connected between bus 4 and reference. Following the procedure of Example 8.2, calculate the voltage *changes* at each of the buses due to the capacitor.

Solution:

The equivalent circuit is:



The capacitive current I_c is $0.84/j(1.1-5.4) = -j0.2$. The voltage changes are then:

$$\Delta \mathbf{V} = \mathbf{Z_{BUS}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ j0.2 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.12 \\ 0.04 \\ 0.22 \end{bmatrix}$$

(8.3) Modify $\mathbf{Z_{BUS}}$ of Prob. 8.1 to include a capacitor of reactance 5.4 per-unit connected between bus 4 and reference and then calculate the new bus voltages using the modified $\mathbf{Z_{BUS}}$. Check your answers using the results of Probs. 8.1 and 8.2.

Solution: Z_{BUS} is augmented to add the shunt capacitor:

$$j\left[\begin{array}{cccc|c} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 & 1.1 \\ \hline 0.2 & 0.6 & 0.2 & 1.1 & -4.3 \end{array}\right]$$

After kron reduction, Z_{BUS} is given by:

$$j\begin{bmatrix} 0.20930 & 0.22791 & 0.20930 & 0.25116 \\ 0.22791 & 0.68372 & 0.22791 & 0.75349 \\ 0.20930 & 0.22791 & 0.80930 & 0.25116 \\ 0.25116 & 0.75349 & 0.25116 & 1.38140 \end{bmatrix}$$

And the voltages are:

$$j\begin{bmatrix} 0.20930 & 0.22791 & 0.20930 & 0.25116 \\ 0.22791 & 0.68372 & 0.22791 & 0.75349 \\ 0.20930 & 0.22791 & 0.80930 & 0.25116 \\ 0.25116 & 0.75349 & 0.25116 & 1.38140 \end{bmatrix}\begin{bmatrix} -j5.9 \\ j0.1 \\ j0.2 \\ j0.2 \end{bmatrix} = \begin{bmatrix} 1.12 \\ 1.08 \\ 1.00 \\ 1.08 \end{bmatrix}$$

Which is the sum of V and ΔV from Probs. 8.1 and 8.2.

(8.4) Modify the Z_{BUS} determined in Example 8.4 for the circuit of Fig. 8.8 by adding a new node connected to Bus 3 through an impedance of $j0.5$ per-unit.

Solution: Let the new node be designated by node q. Z_{BUS} becomes:

$$\begin{array}{cc} & \begin{array}{ccccc} \text{①} & \text{②} & \text{③} & \text{④} & q \end{array} \\ j & \begin{bmatrix} 0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.53340 \\ 0.60992 & 0.73190 & 0.64008 & 0.69659 & 0.64008 \\ 0.53340 & 0.64008 & 0.71660 & 0.66951 & 0.71660 \\ 0.58049 & 0.69659 & 0.66951 & 0.76310 & 0.66951 \\ 0.53340 & 0.64008 & 0.71660 & 0.66951 & 1.21660 \end{bmatrix} \end{array}$$

(8.5) Modify the Z_{BUS} determined in Example 8.4 by adding a branch of impedance $j0.2$ between buses 1 and 4 of the circuit of Fig. 8.8.

Solution:
Z_{BUS} is augmented as follows:

$$\begin{array}{cc} & \begin{array}{ccccc} \text{①} & \text{②} & \text{③} & \text{④} & q \end{array} \\ j & \left[\begin{array}{cccc|c} 0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.13611 \\ 0.60992 & 0.73190 & 0.64008 & 0.69659 & -0.08667 \\ 0.53340 & 0.64008 & 0.71660 & 0.66951 & -0.13611 \\ 0.58049 & 0.69659 & 0.66951 & 0.76310 & -0.18261 \\ \hline 0.13611 & -0.08667 & -0.13611 & -0.18261 & 0.51872 \end{array}\right] \end{array}$$

and by kron reduction, we get:

$$
j\begin{bmatrix} 0.68089 & 0.63266 & 0.56911 & 0.62841 \\ 0.63266 & 0.71742 & 0.61734 & 0.66608 \\ 0.56911 & 0.61734 & 0.68089 & 0.62159 \\ 0.62841 & 0.66608 & 0.62159 & 0.69881 \end{bmatrix}
$$

(8.6) Modify the $\mathbf{Z}_{BUS}$ determined in Example 8.4 by removing the impedance connected between buses 2 and 3 of the circuit of Fig. 8.8.

Solution:
To remove the branch between nodes 2 and 3, we add a second branch of impedance $-j0.4$ between the two nodes. Therefore, $\mathbf{Z}_{BUS}$ is augmented as follows:

$$
\begin{array}{c}
\begin{array}{ccccc} & & & & q \end{array} \\
j\left[\begin{array}{cccc|c} 0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.07652 \\ 0.60992 & 0.73190 & 0.64008 & 0.69659 & 0.09182 \\ 0.53340 & 0.64008 & 0.71660 & 0.66951 & -0.07652 \\ 0.58049 & 0.69659 & 0.66951 & 0.76310 & 0.02708 \\ \hline 0.07652 & 0.09182 & -0.07652 & 0.02708 & -0.23166 \end{array}\right]
\end{array}
$$

and by kron reduction, we get:

$$
j\begin{bmatrix} 0.74188 & 0.64025 & 0.50812 & 0.58943 \\ 0.64025 & 0.76829 & 0.60975 & 0.70732 \\ 0.50812 & 0.60975 & 0.74188 & 0.66057 \\ 0.58943 & 0.70732 & 0.66057 & 0.76627 \end{bmatrix}
$$

(8.7) Find $\mathbf{Z}_{BUS}$ for the circuit of Fig. 7.18 by the $\mathbf{Z}_{BUS}$ building algorithm discussed in Sec. 8.4. Assume there is no mutual coupling between branches.

Solution:

$$0\text{-}1 \quad \begin{array}{c} \text{①} \\ j[1] \end{array}$$

$$1\text{-}2 \quad j \begin{array}{cc} \text{①} & \text{②} \end{array} \atop \begin{bmatrix} 1 & 1 \\ 1 & 1.4 \end{bmatrix}$$

$$1\text{-}3 \quad j \begin{array}{ccc} \text{①} & \text{②} & \text{③} \end{array} \atop \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.4 & 1 \\ 1 & 1 & 1.5 \end{bmatrix}$$

$$2\text{-}3 \quad j \begin{array}{cccc} \text{①} & \text{②} & \text{③} & \text{ⓠ} \end{array} \atop \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1.4 & 1 & 0.4 \\ 1 & 1 & 1.5 & -0.5 \\ \hline 0 & 0.4 & -0.5 & 1.15 \end{array}\right] ;$$

After kron reduction:

$$j \begin{array}{ccc} \text{①} & \text{②} & \text{③} \end{array} \atop \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.26087 & 1.17391 \\ 1 & 1.17391 & 1.28261 \end{bmatrix}$$

$$3\text{-}4 \quad j \begin{array}{cccc} \text{①} & \text{②} & \text{③} & \text{④} \end{array} \atop \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.26087 & 1.17391 & 1.17391 \\ 1 & 1.17391 & 1.28261 & 1.28261 \\ 1 & 1.17391 & 1.28261 & 1.40761 \end{bmatrix}$$

$$
0\text{-}5 \quad j \begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
1 & 1 & 1 & 1 & 0 \\
1 & 1.26087 & 1.17391 & 1.17391 & 0 \\
1 & 1.17391 & 1.28261 & 1.28261 & 0 \\
1 & 1.17391 & 1.28261 & 1.40761 & 0 \\
0 & 0 & 0 & 0 & 1.25
\end{array}\right]
\end{array};
$$

$$
2\text{-}5 \quad j \begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & q \end{array} \\
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1.26087 & 1.17391 & 1.17391 & 0 & 1.26087 \\
1 & 1.17391 & 1.28261 & 1.28261 & 0 & 1.17391 \\
1 & 1.17391 & 1.28261 & 1.40761 & 0 & 1.17391 \\
0 & 0 & 0 & 0 & 1.25 & -1.25 \\
\hline
1 & 1.26087 & 1.17391 & 1.17391 & -1.25 & 2.71087
\end{array}\right]
\end{array}
$$

By kron reduction:

$$
j \begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
0.63111 & 0.53488 & 0.56696 & 0.56696 & 0.46111 \\
0.53488 & 0.67442 & 0.62791 & 0.62791 & 0.58140 \\
0.56696 & 0.62791 & 0.77426 & 0.77426 & 0.54130 \\
0.56696 & 0.62791 & 0.77426 & 0.89926 & 0.54130 \\
0.46111 & 0.58140 & 0.54130 & 0.54130 & 0.67362
\end{array}\right]
\end{array}
$$

$$
4\text{-}5 \quad j \begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & q \end{array} \\
\left[\begin{array}{ccccc|c}
0.63111 & 0.53488 & 0.56696 & 0.56696 & 0.46111 & 0.10585 \\
0.53488 & 0.67442 & 0.62791 & 0.62791 & 0.58140 & 0.04651 \\
0.56696 & 0.62791 & 0.77426 & 0.77426 & 0.54130 & 0.23296 \\
0.56696 & 0.62791 & 0.77426 & 0.89926 & 0.54130 & 0.35796 \\
0.46111 & 0.58140 & 0.54130 & 0.54130 & 0.67362 & -0.13232 \\
\hline
0.10585 & 0.04651 & 0.23296 & 0.35796 & -0.13232 & 0.99028
\end{array}\right]
\end{array}
$$

After final kron reduction, $\mathbf{Z}_{BUS}$ becomes:

$$\begin{array}{cc} & \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \\ j & \begin{bmatrix} 0.61980 & 0.52991 & 0.54206 & 0.52870 & 0.47525 \\ & 0.67224 & 0.61697 & 0.61110 & 0.58761 \\ & & 0.71946 & 0.69005 & 0.57243 \\ & \text{Symmetric} & & 0.76987 & 0.58913 \\ & & & & 0.65594 \end{bmatrix} \end{array}$$

(8.8) For the reactance network of Fig. 8.14, find
(a) $\mathbf{Z}_{BUS}$ by direct formation
(b) The voltage at each bus,
(c) The current drawn by a capacitor having a reactance of 5.0 per unit connected from bus 3 to neutral,
(d) the change in voltage at each bus when the capacitor is connected to bus 3, and
(e) The voltage at each bus after connecting the capacitor. The magnitude and angle of each of the generated voltages is assumed to be constant.

Solution:
(a)

$$\begin{array}{cc} & \begin{array}{ccc} ① & ② & ③ \end{array} \\ \mathbf{Z}_{BUS} = j & \begin{bmatrix} 0.411 & 0.310 & 0.354 \\ 0.310 & 0.446 & 0.333 \\ 0.354 & 0.333 & 0.450 \end{bmatrix} \end{array}$$

(b)

$$\mathbf{V} = \mathbf{Z}_{BUS} \begin{bmatrix} 1.6\angle -90^\circ \\ 1.2\angle -60^\circ \\ 0 \end{bmatrix} = \begin{bmatrix} 0.980 + j0.186 \\ 0.959 + j0.268 \\ 0.912 + j0.200 \end{bmatrix} = \begin{bmatrix} 0.997\angle 10.75^\circ \\ 0.996\angle 15.61^\circ \\ 0.934\angle 12.37^\circ \end{bmatrix}$$

(c) The capacitive current is determined by using $Z_{th} = Z_{33}$ and V_3:

$$I_C = \frac{0.934\angle 12.37^\circ}{j0.450 - j5.0} = 0.205\angle 102.37^\circ \text{ p.u.}$$

(d) The changes in bus voltages due to I_C are:

$$\Delta V = Z_{BUS}\,\Delta I$$

$$= j\begin{bmatrix} 0.411 & 0.310 & 0.354 \\ 0.310 & 0.446 & 0.333 \\ 0.354 & 0.333 & 0.450 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -0.205\angle 102.37^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 0.0726\angle 12.37^\circ \\ 0.0683\angle 12.37^\circ \\ 0.0923\angle 12.37^\circ \end{bmatrix}$$

(e) The resulting voltages are:

$$V_{new} = V + \Delta V$$

$$= \begin{bmatrix} 0.997\angle 10.75^\circ \\ 0.996\angle 15.61^\circ \\ 0.934\angle 12.37^\circ \end{bmatrix} + \begin{bmatrix} 0.0726\angle 12.37^\circ \\ 0.0683\angle 12.37^\circ \\ 0.0923\angle 12.37^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1.070\angle 10.86^\circ \\ 1.064\angle 15.41^\circ \\ 1.026\angle 12.37^\circ \end{bmatrix}$$

(8.9) Find Z_{BUS} for the three-bus circuit of Fig. 8.15 by using the Z_{BUS} building algorithm of Sec. 8.4.

Solution:

$$\text{0-1} \quad \begin{array}{c} ① \\ j\begin{bmatrix} 1.0 \end{bmatrix} \end{array} \qquad \text{0-2} \quad \begin{array}{c} \begin{array}{cc} ① & ② \end{array} \\ j\begin{bmatrix} 1.0 & 0 \\ 0 & 1.25 \end{bmatrix} \end{array}$$

$$\text{1-3} \quad \begin{array}{c} \begin{array}{ccc} ① & ② & ③ \end{array} \\ j\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.25 & 1.25 \\ 0 & 1.25 & 1.3 \end{bmatrix} \end{array}$$

$$\text{1-2} \quad \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ⓠ \end{array} \\ j\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1.25 & 1.25 & -1.25 \\ 0 & 1.25 & 1.3 & -1.25 \\ \hline 1 & -1.25 & -1.25 & 2.45 \end{array}\right] \end{array}$$

After kron reduction, Z_{BUS} is given by:

$$\begin{array}{c} \begin{array}{ccc} ① & ② & ③ \end{array} \\ j\begin{bmatrix} 0.5918 & 0.5102 & 0.5102 \\ 0.5102 & 0.6122 & 0.6122 \\ 0.5102 & 0.6122 & 0.6622 \end{bmatrix} \end{array}$$

(8.10) Find $\mathbf{Z}_{\text{BUS}}$ for the four-bus circuit of Fig. 7.12 which has per-unit *admittances* as marked.

Solution:

$\mathbf{Z}_{\text{BUS}}$ can be found by inverting the $\mathbf{Y}_{\text{BUS}}$ given in the text. $\mathbf{Z}_{\text{BUS}}$ =

$$j\begin{bmatrix} 0.7313 & 0.6914 & 0.6132 & 0.6368 \\ 0.6914 & 0.7197 & 0.6082 & 0.6418 \\ 0.6132 & 0.6082 & 0.6989 & 0.5511 \\ 0.6368 & 0.6418 & 0.5511 & 0.6989 \end{bmatrix}$$

(8.11) The three-bus circuit of Fig. 8.15 has per-unit *reactances* as marked. The symmetrical $\mathbf{Y}_{\text{BUS}}$ for the circuit has the triangular factors

$$\mathbf{L} = \begin{bmatrix} -j6.0 & \cdot & \cdot \\ j5.0 & -j21.633333 & \cdot \\ 0 & j20.0 & -j1.510038 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & -0.833333 & 0 \\ \cdot & 1 & -0.924499 \\ \cdot & \cdot & 1 \end{bmatrix}$$

Use $\mathbf{L}$ and $\mathbf{U}$ to calculate

(a) The elements Z_{12}, Z_{23}, and Z_{33} of the system $\mathbf{Z}_{\text{BUS}}$ and

(b) The Thévénin impedance $Z_{th,13}$ looking into the circuit of Fig. 8.15 between buses 1 and 3.

Solution:

(a) Using the method of section 8.5, the first column of $\mathbf{Z}_{BUS}$ is found from solution of:

$$\mathbf{L}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{U}\begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Solution of vector X gives:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} j0.1667 \\ j0.0385 \\ j0.5102 \end{bmatrix}$$

Solving the second equation involving $\mathbf{U}$ for column one of $\mathbf{Z}_{BUS}$:

$$\begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \end{bmatrix} = \begin{bmatrix} j0.5918 \\ j0.5102 \\ j0.5102 \end{bmatrix}$$

And since in this circuit $Z_{12} = Z_{21}$, we have $Z_{12} = j0.5102$. Similarly, Z_{23}, and Z_{33} are found from:

$$\mathbf{L}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U}\begin{bmatrix} Z_{31} \\ Z_{32} \\ Z_{33} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Giving:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ j0.6622 \end{bmatrix}$$

And:

$$\begin{bmatrix} Z_{31} \\ Z_{32} \\ Z_{33} \end{bmatrix} = \begin{bmatrix} j0.5102 \\ j0.6122 \\ j0.6622 \end{bmatrix}$$

(b) The solution proceeds as follows:

$$\mathbf{L}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{U}\begin{bmatrix} Z_1^{(1\text{-}3)} \\ Z_2^{(1\text{-}3)} \\ Z_3^{(1\text{-}3)} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Giving:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} j0.1667 \\ j0.0385 \\ -j0.1520 \end{bmatrix} \quad \text{And consequently:} \quad \begin{bmatrix} Z_1^{(1\text{-}3)} \\ Z_2^{(1\text{-}3)} \\ Z_3^{(1\text{-}3)} \end{bmatrix} = \begin{bmatrix} j0.0816 \\ -j0.1020 \\ -j0.1520 \end{bmatrix}$$

and $Z_{th,13} = Z_1^{(1\text{-}3)} - Z_3^{(1\text{-}3)} = j0.2336$. Note that the difference is taken in the same order, element 1 - element 3.

(8.12) Use the $\mathbf{Y}_{BUS}$ triangular factors of Prob. 8.11 to calculate the Thévénin impedance Z_{22} looking into the circuit of Fig. 8.15 between bus 2 and reference. Check your answer using the solution to Prob. 8.10.

Solution:

$$\mathbf{L}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{U}\begin{bmatrix} Z_{21} \\ Z_{22} \\ Z_{23} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Giving:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ j0.0462 \\ j0.6122 \end{bmatrix} \text{ And consequently: } \begin{bmatrix} Z_{21} \\ Z_{22} \\ Z_{23} \end{bmatrix} = \begin{bmatrix} j0.5102 \\ j0.6122 \\ j0.6122 \end{bmatrix}$$

with $Z_{22} = j0.6122$.
Check: $j1.2 \parallel j1.25 = j0.6122$.

(8.13) The $\mathbf{Y}_{\text{BUS}}$ for the circuit of Fig. 7.12 has triangular factors **L** and **U** given in Example 7.9. Use the triangular factors to calculate the Thévenin impedance $Z_{th,24}$ looking into the circuit of Fig. 7.12 between buses 2 and 4. Check your answer using the solution of Prob. 8.10.

Solution: Using **L**, solve for the intermediate variables **X**:

$$\begin{bmatrix} -j16.75 & & & \\ j11.75 & -j11.00746 & & \\ j2.5 & j4.25373 & -j3.78305 & \\ j2.5 & j6.75373 & j2.98305 & -j1.43082 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

giving:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ j0.0908 \\ j0.1022 \\ -j0.0571 \end{bmatrix}$$

Solving for the difference of $\mathbf{Z}_{\mathrm{BUS}}$ columns:

$$\begin{bmatrix} 1 & -0.70149 & -0.14925 & -0.14925 \\ & 1 & -0.38644 & -0.61356 \\ & & 1 & -0.78853 \\ & & & 1 \end{bmatrix} \begin{bmatrix} Z_1^{(2\text{-}4)} \\ Z_2^{(2\text{-}4)} \\ Z_3^{(2\text{-}4)} \\ Z_4^{(2\text{-}4)} \end{bmatrix} = \begin{bmatrix} 0 \\ j0.0908 \\ j0.1022 \\ -j0.0571 \end{bmatrix}$$

gives:

$$\begin{bmatrix} Z_1^{(2\text{-}4)} \\ Z_2^{(2\text{-}4)} \\ Z_3^{(2\text{-}4)} \\ Z_4^{(2\text{-}4)} \end{bmatrix} = \begin{bmatrix} j0.0546 \\ j0.0779 \\ j0.0571 \\ -j0.0571 \end{bmatrix}$$

And $Z_{th,24} = = Z_2^{(2\text{-}4)} - Z_4^{(2\text{-}4)} = j0.1350$. Note that the difference is taken in the same order, element 2 - element 4.

Check: Using $\mathbf{Z}_{\mathrm{BUS}}$, $Z_{th,24} = Z_{22} + Z_{44} - 2Z_{24} = j(0.7197 + 0.6989 - 2 \times 0.6418) = j0.1350$.

(8.14) Using the notation of Sec. 8.6, prove that the total reactive power loss is given by the formula $Q_L = \mathbf{I}^T\mathbf{X}_{\mathrm{BUS}}\mathbf{I}^*$.

Solution:

$$S_L = P_L + jQ_L = \mathbf{I}^T\mathbf{Z}_{\mathrm{BUS}}\mathbf{I}^*$$
$$S_L^{*T} = P_L - jQ_L = \mathbf{I}^T\mathbf{Z}_{\mathrm{BUS}}^{*T}\,\mathbf{I}^*$$

Subtracting, we get,

$$\frac{S_L - S_L^{*T}}{2} = jQ_L = \mathbf{I}^T\left[\frac{\mathbf{Z_{BUS}} - \mathbf{Z_{BUS}^{*T}}}{2}\right]\mathbf{I}^*$$

and for symmetric $\mathbf{Z_{BUS}} = \mathbf{R_{BUS}} + j\mathbf{X_{BUS}}$, we have:

$$Q_L = \mathbf{I}^T[\mathbf{X_{BUS}}]\mathbf{I}^*$$

(8.15) Calculate the total reactive power loss in the system of Fig. 8.13 using Eq. (8.57).

Solution:
The vectors **I** and $\mathbf{Z_{BUS}}$ are given in Prob. 8.1.,

$$S = \mathbf{I}^T\mathbf{Z_{BUS}}\mathbf{I}^* =$$

$$= j\begin{bmatrix} -j0.1 & j0.1 & j0.2 & j0.2 \end{bmatrix}\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix}\begin{bmatrix} j0.1 \\ -j0.1 \\ -j0.2 \\ -j0.2 \end{bmatrix}.$$

$$= j0.152$$

(8.16) Using the procedure discussed in Sec. 8.6, modify the $\mathbf{Z_{BUS}}$ determined in Example 8.4 to reflect the choice of bus 2 of Fig. 8.8 as the reference.

Solution:

The transformation C that changes the reference from bus 0 to bus 2 is:

$$
\mathbf{C} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array}
\begin{array}{c}
\begin{array}{cccc} ① & ⓪ & ③ & ④ \end{array} \\
\begin{bmatrix} 1 & & & \\ -1 & -1 & -1 & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}
\end{array}
$$

$$
\mathbf{Z}_{BUS,new} = \mathbf{C}^T \mathbf{Z}_{BUS} \mathbf{C}^* =
$$

$$
\begin{bmatrix} 1 & -1 & & \\ & -1 & & \\ & -1 & 1 & \\ & -1 & & 1 \end{bmatrix}
\begin{bmatrix}
j0.71660 & j0.60992 & j0.53340 & j0.58049 \\
j0.60992 & j0.73190 & j0.64008 & j0.69659 \\
j0.53340 & j0.64008 & j0.71660 & j0.66951 \\
j0.58049 & j0.69659 & j0.66951 & j0.76310
\end{bmatrix}
\begin{bmatrix} 1 & & & \\ -1 & -1 & -1 & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}
$$

$$
= \begin{array}{c}
\begin{array}{cccc} ① & ⓪ & ③ & ④ \end{array} \\
\begin{bmatrix}
j0.22866 & j0.12198 & j0.01530 & j0.00588 \\
j0.12198 & j0.73190 & j0.09182 & j0.03531 \\
j0.01530 & j0.09182 & j0.16834 & j0.06474 \\
j0.00588 & j0.03531 & j0.06474 & j0.10182
\end{bmatrix}
\end{array}
$$

(8.17)

(*a*) Find $\mathbf{Z}_{BUS}$ for the network of Fig. 8.13 using node 5 as reference. Change the reference from node 5 to node 4 and determine the new $\mathbf{Z}_{BUS}$ of the network using Eq. (8.60). Use the numerical values of the load currents I_{L1} of Prob. 8.1 to determine $\mathbf{I}_{new}$ by Eq. (8.55) and $\mathbf{V}_{new}$ by Eq. (8.56).

(b) Change the $\mathbf{Z}_{BUS}$ reference from node 4 back to node 5, using Eq. (8.63), determine the voltages at buses 1 and 4 relative to node 5. What are the values of these bus voltages with respect to the ground reference of Fig. 8.13?

Solution:

(a) From Prob. (8.1),

$$\mathbf{Z}_{BUS} = j\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix}$$

The transformation is:

$$C = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ⑤ \end{array} \\ \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

$$\mathbf{Z}_{BUS,new} = \mathbf{C}^T \mathbf{Z}_{BUS} \mathbf{C}^* =$$

$$= j\begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & -1 \end{bmatrix}\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} ① & ② & ③ & ⑤ \end{array} \\
= j & \begin{bmatrix} 0.9 & 0.5 & 0.9 & 0.9 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.9 & 0.5 & 1.5 & 0.9 \\ 0.9 & 0.5 & 0.9 & 1.1 \end{bmatrix}
\end{array}
$$

The new current vector must be determined from $\mathbf{I}=\mathbf{C}\mathbf{I}_{new}$,

$$
\begin{bmatrix} I_{1,old} \\ I_{2,old} \\ I_{3,old} \\ I_{4,old} \end{bmatrix} = j\begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_{1,new} \\ I_{2,new} \\ I_{3,new} \\ I_{5,new} \end{bmatrix}
$$

giving $\mathbf{I}_{new}^{T} = j(0.1 \quad 0.1 \quad 0.2 \quad -0.6)$. And $\mathbf{V}_{new}$, the voltages with respect to node 4, are obtained from $\mathbf{Z}_{BUS,new}\mathbf{I}_{new}$:

$$
\begin{bmatrix} V_{1,new} \\ V_{2,new} \\ V_{3,new} \\ V_{4,new} \end{bmatrix} = j\begin{bmatrix} 0.9 & 0.5 & 0.9 & 0.9 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.9 & 0.5 & 1.5 & 0.9 \\ 0.9 & 0.5 & 0.9 & 1.1 \end{bmatrix} \begin{bmatrix} j0.1 \\ j0.1 \\ j0.2 \\ -j0.6 \end{bmatrix} = \begin{bmatrix} 0.22 \\ 0.1 \\ 0.1 \\ 0.34 \end{bmatrix}
$$

(b) $\mathbf{Z}_{BUS}$ is given by:

$$
\mathbf{Z}_{BUS} = j\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix}
$$

And the new voltages are given from solution of $V_{new} = C^{T*}V$; V comes from part (a):

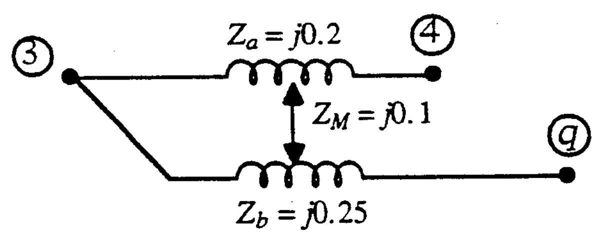
$$V_{new} = \begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} 0.22 \\ 0.1 \\ 0.1 \\ 0.34 \end{bmatrix} = \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix}$$

And the voltage with respect to a ground reference is:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = 1.2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix} = \begin{bmatrix} 1.08 \\ 0.96 \\ 0.96 \\ 0.86 \end{bmatrix}$$

(8.18) A new branch having an impedance of $j0.25$ per unit is connected between nodes 3 and 4 of the circuit of Fig. 8.8 in parallel with the existing impedance of $j0.2$ per unit between the same two nodes. These two branches have mutual impedance of $j0.1$ per unit. Modify the Z_{BUS} determined in Example 8.4 to account for the addition of the new branch.

Solution:
First, add the mutually coupled line forming a new bus q:



$$Z_{qj} = Z_{pj} - \frac{Z_M}{Z_a}(Z_{mj} - Z_{nj})$$

$$Z_{qq} = Z_{pq} - \frac{Z_M}{Z_a}(Z_{mq} - Z_{nq}) - \left(\frac{Z_M^2}{Z_a} - Z_b\right)$$

$$Z_{q1} = j0.53340 - \frac{j0.1}{j0.2}(j0.53340 - j0.58049) = j0.55695$$
$$Z_{q2} = j0.64008 - \frac{j0.1}{j0.2}(j0.64008 - j0.69659) = j0.66834$$
$$Z_{q3} = j0.71660 - \frac{j0.1}{j0.2}(j0.71660 - j0.66951) = j0.69306$$
$$Z_{q4} = j0.66951 - \frac{j0.1}{j0.2}(j0.66951 - j0.76310) = j0.71631$$
$$Z_{qq} = j0.69306 - \frac{j0.1}{j0.2}(j0.69306 - j0.71631) - \left(\frac{(j0.1)^2}{j0.2} - j0.25\right) = j0.90469$$

The augmented matrix is:

$$\begin{array}{c} \\ \\ \\ = j \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{①} & \text{②} & \text{③} & \text{④} & \text{ⓠ} \end{array} \\
\left[\begin{array}{cccc|c}
0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.55695 \\
0.60992 & 0.73190 & 0.64008 & 0.69659 & 0.66834 \\
0.53340 & 0.64008 & 0.71660 & 0.66951 & 0.69306 \\
0.58049 & 0.69659 & 0.66951 & 0.76310 & 0.71631 \\
\hline
0.55695 & 0.66834 & 0.69306 & 0.71631 & 0.90469
\end{array}\right]
\end{array}$$

To parallel nodes 4 and q, row and column q is subtracted from row and column 4 to give:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & q - 4 \end{array} \\
j & \left[\begin{array}{cccc|c}
0.71660 & 0.60992 & 0.53340 & 0.58049 & -0.02354 \\
0.60992 & 0.73190 & 0.64008 & 0.69659 & -0.02825 \\
0.53340 & 0.64008 & 0.71660 & 0.66951 & 0.02355 \\
0.58049 & 0.69659 & 0.66951 & 0.76310 & -0.04679 \\
\hline
-0.02354 & -0.02825 & 0.02355 & -0.04679 & 0.23517
\end{array}\right]
\end{array}
$$

After kron reduction, we have

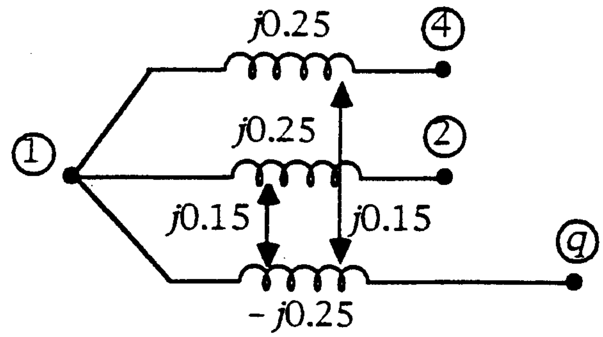
$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
j & \begin{bmatrix}
0.71424 & 0.60709 & 0.53576 & 0.57581 \\
0.60709 & 0.72851 & 0.64291 & 0.69097 \\
0.53576 & 0.64291 & 0.71424 & 0.67420 \\
0.57581 & 0.69097 & 0.67420 & 0.75379
\end{bmatrix}
\end{array}
$$

(8.19) Derive Eqs. (8.95) and (8.96).

Solution:
These equations are a special case of the equations developed in Prob. 8.23.

(8.20) Modify the Z_{BUS} determined in Example 8.7 to remove the branch between buses 1 and 2 already coupled by mutual impedance $j0.15$ per unit to the branch between buses 1 and 4.

Solution:



$$\begin{bmatrix} Y_a & Y_M \\ Y_M & Y_B \end{bmatrix} = \begin{bmatrix} Z_a & Z_M \\ Z_M & Z_B \end{bmatrix}^{-1} = \begin{bmatrix} j0.25 & j0.15 \\ j0.15 & j0.25 \end{bmatrix}^{-1} = \begin{bmatrix} -j6.25 & j3.75 \\ j3.75 & -j6.25 \end{bmatrix}$$

$$Z_{qj} = Z_{1j} - j0.15[\ -j6.25 \quad j3.75\ ]\begin{bmatrix} Z_{1j} - Z_{4j} \\ Z_{1j} - Z_{2j} \end{bmatrix}$$

$$Z_{qq} = Z_{pq} - j0.15[\ -j6.25 \quad j3.75\ ]\begin{bmatrix} Z_{1q} - Z_{4q} \\ Z_{1q} - Z_{2q} \end{bmatrix} - (\ -j6.25(j0.15)^2 + j0.25\ )$$

Thus, we have:

$$Z_{q1} = j0.69890 - j0.15[\ -j6.25 \quad j3.75\ ]\begin{bmatrix} j0.69890 - j0.60822 \\ j0.69890 - j0.61323 \end{bmatrix} = j0.66208$$

$$Z_{q2} = j0.61323 - j0.15[\ -j6.25 \quad j3.75\ ]\begin{bmatrix} j0.61323 - j0.69140 \\ j0.61323 - j0.73128 \end{bmatrix} = j0.62011$$

$$Z_{q3} = j0.55110 - j0.15[\ -j6.25 \quad j3.75\ ]\begin{bmatrix} j0.55110 - j0.64178 \\ j0.55110 - j0.63677 \end{bmatrix} = j0.58792$$

$$Z_{q4} = j0.60822 - j0.15[\ -j6.25 \quad j3.75\ ]\begin{bmatrix} j0.60822 - j0.71966 \\ j0.60822 - j0.69140 \end{bmatrix} = j0.66591$$

$$Z_{qq} = j0.66208 - j0.15[\ -j6.25 \quad j3.75\ ]\begin{bmatrix} j0.66208 - j0.66591 \\ j0.66208 - j0.62011 \end{bmatrix}$$
$$- (\ -j6.25(j0.15)^2 + j0.25\ ) = j0.29865$$

The augmented matrix is:

$$\begin{array}{cc}
 & \begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad q \quad \end{array} \\
j & \left[\begin{array}{cccc|c}
0.69890 & 0.61323 & 0.55110 & 0.60822 & 0.66208 \\
0.61323 & 0.73128 & 0.63677 & 0.69140 & 0.62011 \\
0.55110 & 0.63677 & 0.69890 & 0.64178 & 0.58792 \\
0.60822 & 0.69140 & 0.64178 & 0.71966 & 0.66591 \\
\hline
0.66208 & 0.62011 & 0.58792 & 0.66591 & 0.29865
\end{array}\right]
\end{array}$$

Connecting buses 2 and q, we have:

$$\begin{array}{cc}
 & \begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad q - 2 \quad \end{array} \\
j & \left[\begin{array}{cccc|c}
0.69890 & 0.61323 & 0.55110 & 0.60822 & 0.04885 \\
0.61323 & 0.73128 & 0.63677 & 0.69140 & -0.11117 \\
0.55110 & 0.63677 & 0.69890 & 0.64178 & -0.04885 \\
0.60822 & 0.69140 & 0.64178 & 0.71966 & -0.02549 \\
\hline
0.04885 & -0.11117 & -0.04885 & -0.02549 & -0.21029
\end{array}\right]
\end{array}$$

After kron reduction:

$$\begin{array}{cc}
 & \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\
j & \left[\begin{array}{cccc}
0.71025 & 0.58741 & 0.53975 & 0.60230 \\
0.58741 & 0.79005 & 0.66259 & 0.70488 \\
0.53975 & 0.66259 & 0.71025 & 0.64770 \\
0.60230 & 0.70488 & 0.64770 & 0.72275
\end{array}\right]
\end{array}$$

(8.21) Assume that the two branches 1-3 and 2-3 in the circuit of Fig. 7.18 are the only mutually coupled branches (as indicated by the dots) with a mutual impedance of $j0.15$ per unit between them. Find $\mathbf{Z}_{BUS}$ for the circuit by the $\mathbf{Z}_{BUS}$ building algorithm.

Solution:

0-1:
$$\begin{array}{c} (1) \\ j\,[1] \end{array}$$

1-3:
$$\begin{array}{cc} (1) & (3) \end{array}$$
$$j\begin{bmatrix} 1 & 1 \\ 1 & 1.5 \end{bmatrix}$$

3-2: Using the formula given in the text:

$$Z_{21} = j1 - \frac{j0.15}{j0.5}(j1 - j1) = j1$$

$$Z_{23} = j1.5 - \frac{j0.15}{j0.5}(j1.5 - j1) = j1.35$$

$$Z_{22} = j1.35 - \frac{j0.15}{j0.5}(j1.35 - j1) + j0.205 = j1.45$$

$$\begin{array}{ccc} (1) & (3) & (2) \end{array}$$
$$j\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.35 \\ 1 & 1.35 & 1.45 \end{bmatrix}$$

1-2: Form the augmented matrix:

$$\begin{array}{cc}
 & \begin{array}{cccc} 1 & 3 & 2 & \end{array} \\
j & \left[\begin{array}{ccc|c}
1 & 1 & 1 & 0 \\
1 & 1.5 & 1.35 & -0.35 \\
1 & 1.35 & 1.45 & -0.45 \\
\hline
0 & -0.35 & -0.45 & 0.85
\end{array}\right]
\end{array}$$

and by kron reduction we get:

$$\begin{array}{cc}
 & \begin{array}{ccc} 1 & 3 & 2 \end{array} \\
j & \left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1.35588 & 1.16471 \\
1 & 1.16471 & 1.21176
\end{array}\right]
\end{array}$$

3-4:

$$\begin{array}{cc}
 & \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} \\
j & \left[\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088
\end{array}\right]
\end{array}$$

4-5:

$$\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 3 & 2 & 4 & 5 \end{array} \\
j & \left[\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088
\end{array}\right]
\end{array}$$

0-5:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 3 & 2 & 4 & 5 & \end{array} \\
j & \left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 & 1.48088 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 1.98088 \\
\hline
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 3.23088
\end{array}\right]
\end{array}
$$

and after kron reduction:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 3 & 2 & 4 & 5 \end{array} \\
j & \left[\begin{array}{ccccc}
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 \\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 \\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 \\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 \\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639
\end{array}\right]
\end{array}
$$

2-5: Using Case 4:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 3 & 2 & 4 & 5 & \end{array} \\
j & \left[\begin{array}{ccccc|c}
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 & 0.25262 \\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 & 0.15134 \\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 & 0.34127 \\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 & 0.05792 \\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639 & -0.31577 \\
\hline
0.25262 & 0.15134 & 0.34127 & 0.05792 & -0.31577 & 0.85704
\end{array}\right]
\end{array}
$$

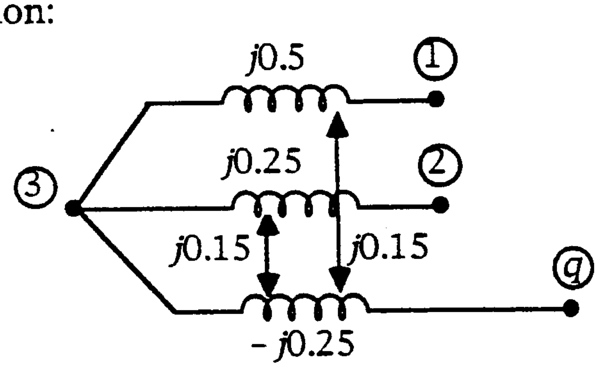
and after kron reduction:

$$
\begin{array}{cccccc}
 & \textcircled{1} & \textcircled{3} & \textcircled{2} & \textcircled{4} & \textcircled{5} \\
j & \multicolumn{5}{c}{\begin{bmatrix}
0.61603 & 0.53573 & 0.53892 & 0.52458 & 0.47997 \\
 & 0.76015 & 0.61566 & 0.72418 & 0.58034 \\
 & & 0.65600 & 0.60780 & 0.57636 \\
 & \text{Symmetric} & & 0.79821 & 0.59428 \\
 & & & & 0.65005
\end{bmatrix}}
\end{array}
$$

(Note node ordering.)

(8.22) Modify the Z_{BUS} obtained in Prob. 8.21 to remove branch 2-3 which is coupled to branch 1-3 through a mutual impedance of $j0.15$ per unit.

Solution:



$$\begin{bmatrix} Y_a & Y_M \\ Y_M & Y_B \end{bmatrix} = \begin{bmatrix} Z_a & Z_M \\ Z_M & Z_B \end{bmatrix}^{-1} = \begin{bmatrix} j0.5 & j0.15 \\ j0.15 & j0.25 \end{bmatrix}^{-1} = \begin{bmatrix} -j2.43902 & j1.46341 \\ j1.46341 & -j4.87805 \end{bmatrix}$$

$$Z_{q1} = j0.53573 - j0.15[\,-j2.43902 \quad j1.46341\,]\begin{bmatrix} j0.53573 - j0.61603 \\ j0.53573 - j0.53892 \end{bmatrix} = j0.56441$$

$$Z_{q2} = j0.61566 - j0.15[\,-j2.43902 \quad j1.46341\,]\begin{bmatrix} j0.61566 - j0.53892 \\ j0.61566 - j0.65600 \end{bmatrix} = j0.57873$$

$$Z_{q3} = j0.76015 - j0.15[\,-j2.43902 \quad j1.46341\,]\begin{bmatrix} j0.76015 - j0.53573 \\ j0.76015 - j0.61566 \end{bmatrix} = j0.70976$$

$$Z_{q4} = j0.72418 - j0.15[\,-j2.43902 \quad j1.46341\,]\begin{bmatrix} j0.72418 - j0.52458 \\ j0.72418 - j0.60780 \end{bmatrix} = j0.67670$$

$$Z_{q5} = j0.58034 - j0.15[\,-j2.43902 \quad j1.46341\,]\begin{bmatrix} j0.58034 - j0.47997 \\ j0.58034 - j0.57636 \end{bmatrix} = j0.54449$$

$$Z_{qq} = j0.70976 - j0.15[\,-j2.43902 \quad j1.46341\,]\begin{bmatrix} j0.70976 - j0.56441 \\ j0.70976 - j0.57873 \end{bmatrix}$$
$$-(\,-j2.43902(j0.15)^2 + j0.15\,) = j0.48047$$

The matrix is given by:

$$\begin{array}{cc}
 & \begin{array}{cccccc} \quad 1 \quad & \quad 3 \quad & \quad 2 \quad & \quad 4 \quad & \quad 5 \quad & \quad q \quad \end{array} \\
j & \left[\begin{array}{ccccc|c}
0.61603 & 0.53573 & 0.53892 & 0.52458 & 0.47997 & 0.56441 \\
0.53573 & 0.76015 & 0.61566 & 0.72418 & 0.58034 & 0.70976 \\
0.53892 & 0.61566 & 0.65600 & 0.60780 & 0.57636 & 0.57873 \\
0.52458 & 0.72418 & 0.60780 & 0.79821 & 0.59428 & 0.67670 \\
0.47997 & 0.58034 & 0.57636 & 0.59428 & 0.65005 & 0.54449 \\
\hline
0.56441 & 0.70976 & 0.57873 & 0.67670 & 0.54449 & 0.48047
\end{array}\right]
\end{array}$$

and node q is joined to node 2:

$$
\begin{array}{c}
\begin{array}{cccccc} \text{①} & \text{③} & \text{②} & \text{④} & \text{⑤} & \text{ⓠ - ②} \end{array} \\
j\left[\begin{array}{ccccc|c}
0.61603 & 0.53573 & 0.53892 & 0.52458 & 0.47997 & 0.02549 \\
0.53573 & 0.76015 & 0.61566 & 0.72418 & 0.58034 & 0.09410 \\
0.53892 & 0.61566 & 0.65600 & 0.60780 & 0.57636 & -0.07727 \\
0.52458 & 0.72418 & 0.60780 & 0.79821 & 0.59428 & 0.06890 \\
0.47997 & 0.58034 & 0.57636 & 0.59428 & 0.65005 & -0.03187 \\
\hline
0.02549 & 0.09410 & -0.07727 & 0.06890 & -0.03187 & -0.02099
\end{array}\right]
\end{array}
$$

and after kron reduction:

$$
\begin{array}{c}
\begin{array}{ccccc} \text{①} & \text{③} & \text{②} & \text{④} & \text{⑤} \end{array} \\
j\left[\begin{array}{ccccc}
0.64698 & 0.65000 & 0.44508 & 0.60825 & 0.44127 \\
 & 1.18201 & 0.26925 & 1.03306 & 0.43746 \\
 & & 0.94045 & 0.35416 & 0.69368 \\
 & \text{Symmetric} & & 1.02438 & 0.48967 \\
 & & & & 0.69844
\end{array}\right]
\end{array}
$$

(8.23) In Fig. 8.16 a new bus q is to be connected to an existing bus p through a new branch c. New branch c is mutually coupled to branches a and b, which are already coupled to one another as shown. The primitive impedance matrix defining the self- and mutual impedances of these three mutually coupled branches and its reciprocal, the primitive admittance matrix, have the forms:

$$
\begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix}
$$

To account for the addition of the new bus q, prove that the existing bus impedance matrix of the network must be augmented by a new row q and column q with elements given by:

$$Z_{qi} = Z_{pi} + \frac{1}{Y_{cc}}\begin{bmatrix} Y_{ca} & Y_{ca} \end{bmatrix}\begin{bmatrix} Z_{mi} - Z_{ni} \\ Z_{ji} - Z_{ki} \end{bmatrix}$$

$$Z_{qq} = Z_{pq} + \frac{1}{Y_{cc}} + \frac{1}{Y_{cc}}\begin{bmatrix} Y_{ca} & Y_{ca} \end{bmatrix}\begin{bmatrix} Z_{mq} - Z_{nq} \\ Z_{jq} - Z_{kq} \end{bmatrix}$$

Note that these equations are generalization of Eqs. (8.87) and (8.88).

Solution:
The voltage drop equations for the three mutually coupled branches are:

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix}^{-1}\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix}^{-1}\begin{bmatrix} V_m - V_n \\ V_j - V_k \\ V_p - V_q \end{bmatrix}$$

Since

$$I_q = -I_c = -Y_{ca}(V_m - V_n) - Y_{cb}(V_j - V_k) - Y_{cc}(V_p - V_q)$$

we have

$$V_q = V_p + \frac{1}{Y_{cc}}\begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix}\begin{bmatrix} V_m - V_n \\ V_j - V_k \end{bmatrix} + \frac{1}{Y_{cc}} I_q$$

Using this equation and the expression for V_i^o given in the text, we have for the q^{th} row:

$$V_q^{(0)} + Z_{qq}I_q = V_p^{(0)} + Z_{pq}I_q + \frac{1}{Y_{cc}}\begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix}\begin{bmatrix} V_m^{(0)} - V_n^{(0)} + (Z_{mq} - Z_{nq})I_q \\ V_j^{(0)} - V_k^{(0)} + (Z_{jq} - Z_{kq})I_q \end{bmatrix} + \frac{1}{Y_{cc}} I_q$$

When $I_q = 0$, we have,

$$V_q^{(0)} = V_p^{(0)} + \frac{1}{Y_{cc}}\begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix}\begin{bmatrix} V_m^{(0)} - V_n^{(0)} \\ V_j^{(0)} - V_k^{(0)} \end{bmatrix}$$

Since

$$V_q^{(0)} = \sum_{i=1}^{N} Z_{qi}V_i \qquad i = 1, \ldots, N, q$$

we note that

$$Z_{qi} = Z_{pi} + \frac{1}{Y_{cc}}\begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix}\begin{bmatrix} Z_{mi} - Z_{ni} \\ Z_{ji} - Z_{ki} \end{bmatrix} \qquad \text{for } i = 1, \ldots N$$

Z_{qq} is determined by setting all other currents except I_q equal to zero:

$$Z_{qq} = Z_{pq} + \frac{1}{Y_{cc}}\begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix}\begin{bmatrix} Z_{mq} - Z_{nq} \\ Z_{jq} - Z_{kq} \end{bmatrix} + \frac{1}{Y_{cc}}$$

(8.24) Branch 2-3 of the circuit of Fig. 7.18 is mutually coupled to two branches 1-3 and 2-5 through mutual impedances of $j0.15$ per unit and $j0.1$ per unit, respectively, as indicated by the dots. Using the formula given in Prob. 8.23, find the Z_{BUS} for the circuit by the Z_{BUS} building algorithm.

Solution:

0-1:
$$\begin{array}{c} ① \\ j\,[1] \end{array}$$

1-3:
$$\begin{array}{cc} & \begin{array}{cc} ① & ③ \end{array} \\ j & \begin{bmatrix} 1 & 1 \\ 1 & 1.5 \end{bmatrix} \end{array}$$

3-2: Using the formula given in the text:

$$Z_{21} = j1 - \frac{j0.15}{j0.5}(j1 - j1) = j1$$

$$Z_{23} = j1.5 - \frac{j0.15}{j0.5}(j1.5 - j1) = j1.35$$

$$Z_{22} = j1.35 - \frac{j0.15}{j0.5}(j1.35 - j1) + j0.205 = j1.45$$

$$\begin{array}{cc} & \begin{array}{ccc} ① & ③ & ② \end{array} \\ j & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.35 \\ 1 & 1.35 & 1.45 \end{bmatrix} \end{array}$$

1-2: Form the augmented matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 3 & 2 & \end{array} \\
j & \left[\begin{array}{ccc|c}
1 & 1 & 1 & 0 \\
1 & 1.5 & 1.35 & -0.35 \\
1 & 1.35 & 1.45 & -0.45 \\
\hline
0 & -0.35 & -0.45 & 0.85
\end{array}\right]
\end{array}
$$

and by kron reduction we get:

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 3 & 2 \end{array} \\
j & \left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1.35588 & 1.16471 \\
1 & 1.16471 & 1.21176
\end{array}\right]
\end{array}
$$

3-4:

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} \\
j & \left[\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088
\end{array}\right]
\end{array}
$$

4-5:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 3 & 2 & 4 & 5 \end{array} \\
j & \left[\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088
\end{array}\right]
\end{array}
$$

0-5:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \quad 1 \quad & \quad 3 \quad & \quad 2 \quad & \quad 4 \quad & \quad 5 \quad & \end{array} \\
j & \left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 & 1.48088 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 1.98088 \\
\hline
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 3.23088
\end{array}\right]
\end{array}
$$

and after kron reduction:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \quad 1 \quad & \quad 3 \quad & \quad 2 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\
j & \left[\begin{array}{ccccc}
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 \\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 \\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 \\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 \\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639
\end{array}\right]
\end{array}
$$

2-5: The primitive admittance matrix of the three coupled branches is obtained as:

$$Y = \mathbf{Z}^{-1} = \begin{bmatrix} j0.5 & j0.15 & 0 \\ j0.15 & j0.25 & j0.1 \\ 0 & j0.1 & j0.2 \end{bmatrix}^{-1}$$

$$= j\begin{bmatrix} -2.58065 & 1.93548 & -0.96774 \\ 1.93548 & -6.45161 & 3.22581 \\ -0.96774 & 3.22581 & -j6.61290 \end{bmatrix}$$

From the equations given:

$$Z_{q1} = j0.38689 + \frac{1}{-j6.61290}[\,-j0.96774 \quad j3.22581\,]\begin{bmatrix} j0.58034 - j0.69049 \\ j0.58034 - j0.63951 \end{bmatrix} = j0.39963$$

$$Z_{q2} = j0.45062 + \frac{1}{-j6.61290}[\,-j0.96774 \quad j3.22581\,]\begin{bmatrix} j0.67592 - j0.63951 \\ j0.67592 - j0.79189 \end{bmatrix} = j0.51252$$

$$Z_{q3} = j0.52458 + \frac{1}{-j6.61290}[\,-j0.96774 \quad j3.22581\,]\begin{bmatrix} j0.78687 - j0.58034 \\ j0.78687 - j0.67592 \end{bmatrix} = j0.50068$$

$$Z_{q4} = j0.57294 + \frac{1}{-j6.61290}[\,-j0.96774 \quad j3.22581\,]\begin{bmatrix} j0.73441 - j0.54165 \\ j0.73441 - j0.63086 \end{bmatrix} = j0.55064$$

$$Z_{q5} = j0.76639 + \frac{1}{-j6.61290}[\,-j0.96774 \quad j3.22581\,]\begin{bmatrix} j0.52458 - j0.38689 \\ j0.52458 - j0.45062 \end{bmatrix} = j0.75046$$

$$Z_{qq} = j0.75046 + \frac{1}{-j6.61290}[\,-j0.96774 \quad j3.22581\,]\begin{bmatrix} j0.50068 - j0.39963 \\ j0.50068 - j0.51252 \end{bmatrix}$$
$$+ \frac{1}{-j6.61290} = j0.92224$$

the augmented matrix is

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \quad 1 \quad & \quad 3 \quad & \quad 2 \quad & \quad 4 \quad & \quad 5 \quad & \quad q \quad \end{array} \\
j & \left[\begin{array}{ccccc|c}
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 & 0.39963 \\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 & 0.50068 \\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 & 0.51252 \\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 & 0.55064 \\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639 & 0.75046 \\
\hline
0.39963 & 0.50068 & 0.51252 & 0.55064 & 0.75046 & 0.92224
\end{array}\right]
\end{array}
$$

connecting buses 2 and q, we have:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \quad 1 \quad & \quad 3 \quad & \quad 2 \quad & \quad 4 \quad & \quad 5 \quad & q - 2 \end{array} \\
j & \left[\begin{array}{ccccc|c}
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 & -0.23988 \\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 & -0.17524 \\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 & -0.27937 \\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 & -0.08022 \\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639 & 0.29984 \\
\hline
-0.23988 & -0.17524 & -0.27937 & -0.08022 & 0.29984 & 0.68909
\end{array}\right]
\end{array}
$$

and after kron reduction:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \quad 1 \quad & \quad 3 \quad & \quad 2 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\
j & \left[\begin{array}{ccccc}
0.60699 & 0.51934 & 0.54226 & 0.51372 & 0.49127 \\
 & 0.74231 & 0.60487 & 0.71401 & 0.60083 \\
 & & 0.67863 & 0.59834 & 0.57218 \\
 & \text{Symmetric} & & 0.79278 & 0.60785 \\
 & & & & 0.63592
\end{array}\right]
\end{array}
$$