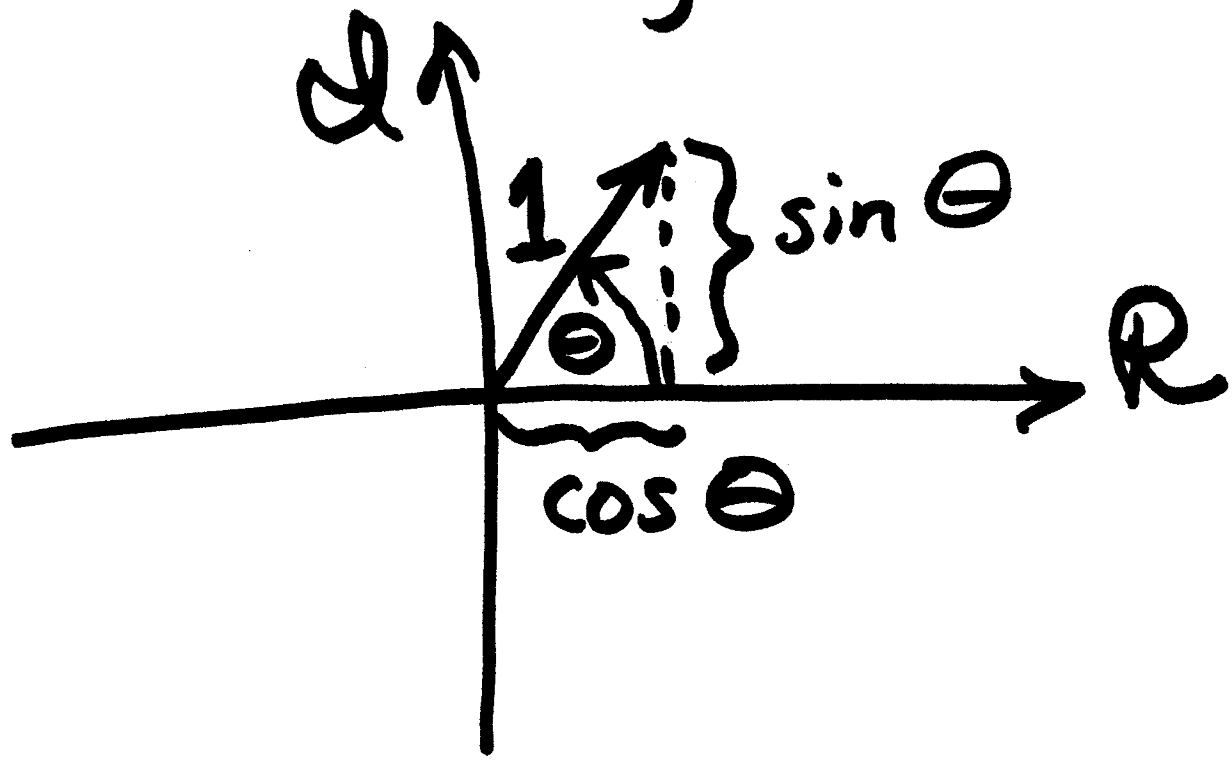
# Euler's Identity

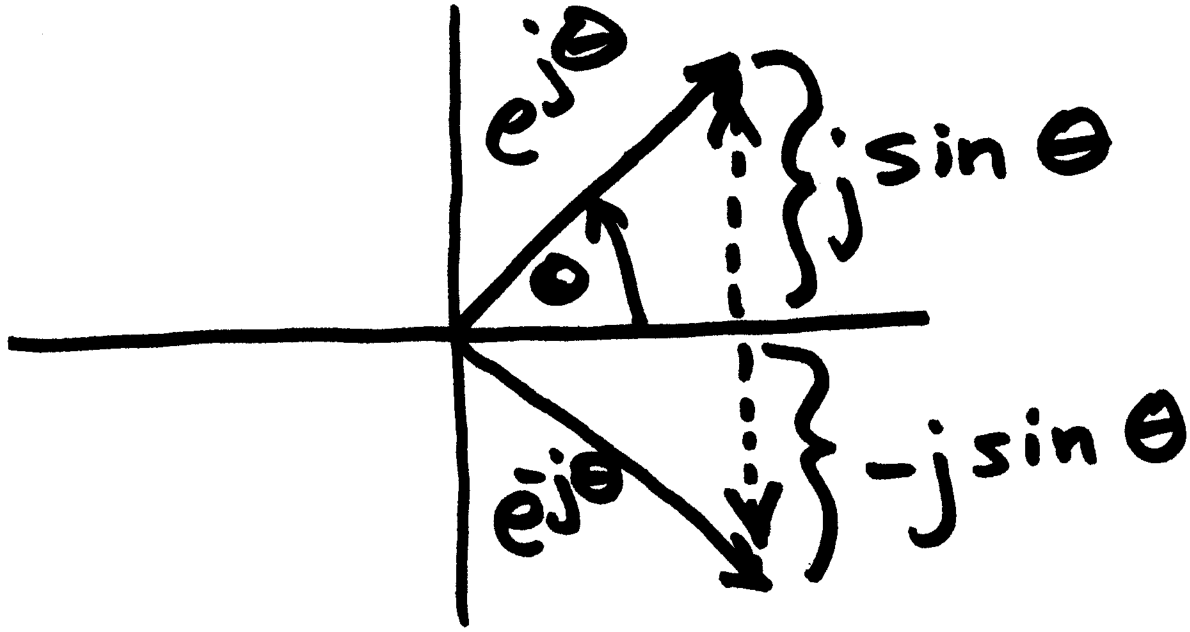


$e^{j\theta}$ = Unit vector at an angle $\theta$.




$$\therefore\ e^{j\theta} = \cos\theta + j\sin\theta \quad (1)$$

$$e^{-j\theta} = \cos\theta - j\sin\theta \quad (2)$$




If we add the 2 eqns

(1) + (2)

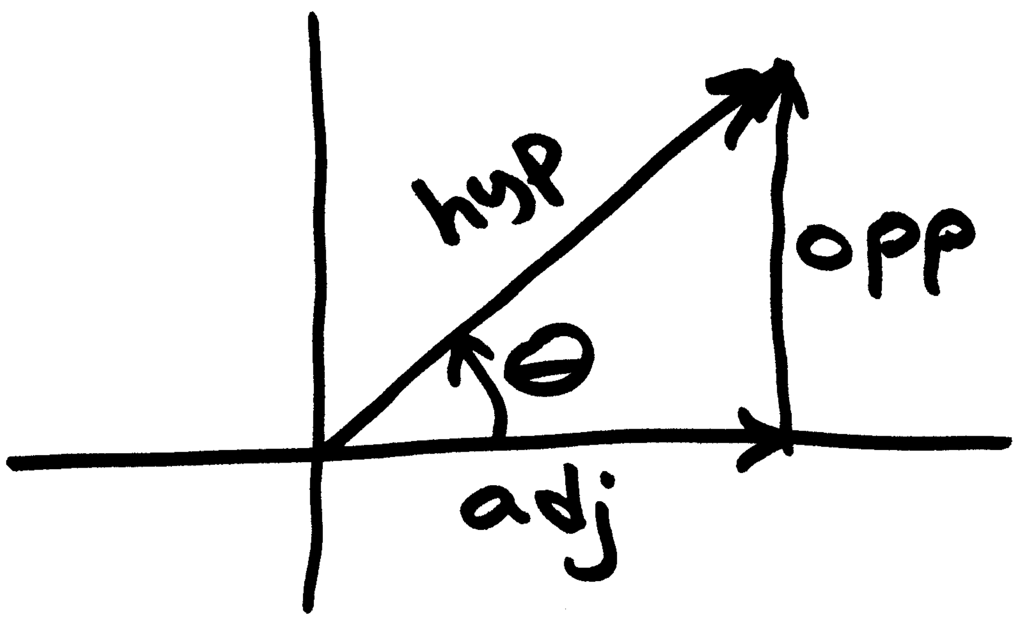
$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Subtracting,

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{j2}}$$

Referring back to figure on preceding page, this matches with the basic trig you learned:




$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$

$$\tan = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta}$$

Hyperbolic Functions are similar exponential functions, where the exponent of e is, in general, a complex number, $z = a + jb$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

$$= \frac{e^{a}e^{jb} - e^{-a}e^{-jb}}{2}$$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

$$= \frac{e^{a}e^{jb} + e^{-a}e^{-jb}}{2}$$

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Examples for real values of $z$:

$z = a + j\cancel{b}$ 0

$z = a + X$

### (1) Definitions

~~A~~ hyperbolic function is a combination of $e^x$ and $e^{-x}$ and is introduced as follows:

$$\text{Hyperbolic sine of } x = \sinh x = \frac{e^x - e^{-x}}{2}$$

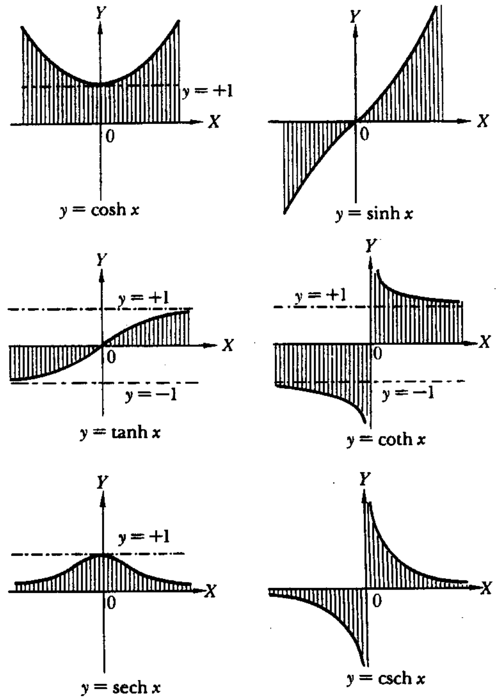
$$\text{Hyperbolic cosine of } x = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\text{Hyperbolic tangent of } x = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{Hyperbolic cotangent of } x = \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{Hyperbolic secant of } x = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\text{Hyperbolic cosecant of } x = \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$




### (2) Relationships

$\cosh^2 x - \sinh^2 x = 1$

$\tanh^2 x + \operatorname{sech}^2 x = 1$

$\coth^2 x - \operatorname{csch}^2 x = 1$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$

$\operatorname{sech} x \cosh x = 1$

$\operatorname{csch} x \sinh x = 1$

$\tanh x \coth x = 1$

$\sinh(-x) = -\sinh x$

$\operatorname{sech}(-x) = \operatorname{sech} x$

$\tanh(-x) = -\tanh x$

$\coth(-x) = -\coth x$

$\cosh(-x) = \cosh x$

$\operatorname{csch}(-x) = -\operatorname{csch} x$

### (3) Limit Values

| $x$ | $\sinh x$ | $\cosh x$ | $\tanh x$ | $\coth x$ | $\operatorname{sech} x$ | $\operatorname{csch} x$ |
|---|---|---|---|---|---|---|
| $-\infty$ | $-\infty$ | $+\infty$ | $-1$ | $-1$ | 0 | 0 |
| $-1$ | $-1.1752$ | $+1.5431$ | $-0.7616$ | $-1.3130$ | $+0.6480$ | $-0.8509$ |
| 0 | 0 | $+1$ | 0 | $\mp\infty$ | $+1$ | $\mp\infty$ |
| $+1$ | $+1.1752$ | $+1.5431$ | $+0.7616$ | $+1.3130$ | $+0.6480$ | $+0.8509$ |
| $+\infty$ | $+\infty$ | $+\infty$ | $+1$ | $+1$ | 0 | 0 |