Finite Element Analysis for Trusses

Preliminary Ideas

Principle of Minimum Total Potential Energy

Given the body and traction forces the displacements are such as to make π a minimum.

$$\pi = U - \int_{Vol} \overline{b} \cdot \overline{u} dv - \int_{Surface} \bar{t} \cdot \overline{u} ds$$
$$U = \text{Strain Energy}$$
$$\overline{b} = \text{Body Forces} \qquad (1.1)$$
$$\bar{t} = \text{Traction Forces}$$
$$\overline{u} = \text{Displacement}$$

$$U = \frac{1}{2} \int_{Vol} (\varepsilon_x \sigma_x + \varepsilon_y \sigma_y + \varepsilon_z \sigma_z + \gamma_{xy} \tau_{xy} + \gamma_{xz} \tau_{xz} + \gamma_{zy} \tau_{zy}) dv \qquad (1.2)$$

Minimization of a quadratic form

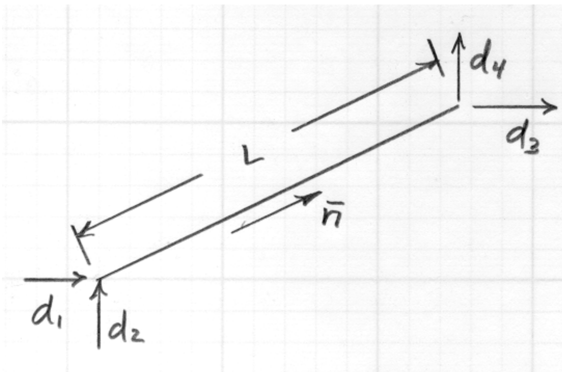
$$\pi = \frac{1}{2} \overline{D}^t [K] \overline{D} - \overline{D}^t \overline{F} + \text{constant} \qquad (1.3)$$

The D_i which make π stationary are given by

$$[K] \overline{D} = \overline{F} \qquad (1.4)$$

Calculation of strain energy in a truss member

Consider the truss element shown below



The elongation (δ) of the truss element is given by:

$$\delta = -n_x d_1 - n_y d_2 + n_x d_3 + n_y d_4 \tag{1.5}$$

In matrix notation we can write this as:

$$\delta = \begin{bmatrix} -n_x & -n_y & +n_x & +n_y \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{bmatrix} -\bar{n}^t & \bar{n}^t \end{bmatrix} \bar{d} = \bar{d}^t \begin{Bmatrix} -\bar{n} \\ \bar{n} \end{Bmatrix} \tag{1.6}$$

We can then write

$$\delta^2 = \bar{d}^t \begin{Bmatrix} -\bar{n} \\ \bar{n} \end{Bmatrix} \begin{bmatrix} -\bar{n}^t & \bar{n}^t \end{bmatrix} \bar{d} \tag{1.7}$$

The strain energy in the bar can then be written as:

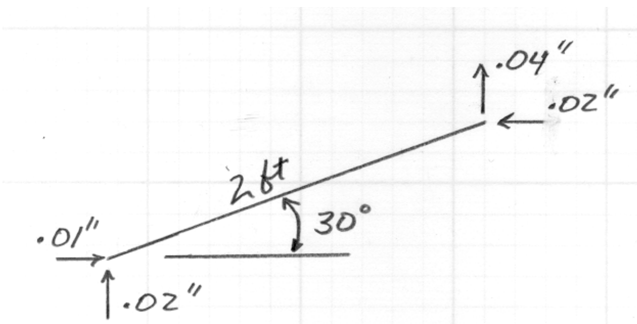
$$U = \frac{1}{2}\left(\frac{AE}{L}\right)\delta^2 = \frac{1}{2}\bar{d}^t [k] \bar{d} \tag{1.8}$$

where $[k]$ is called the element stiffness matrix and is given by:

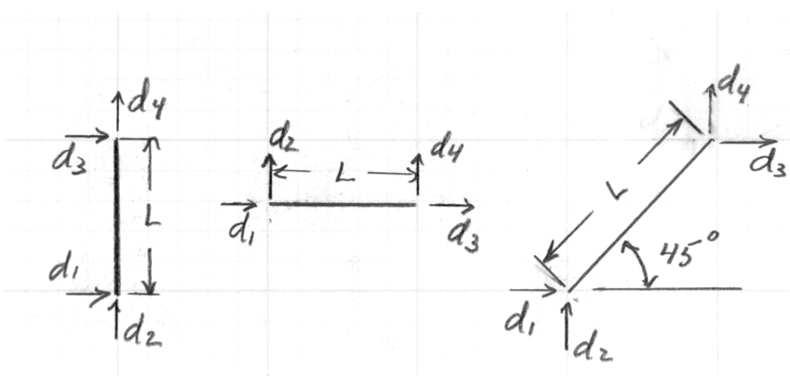
$$[k] = \frac{AE}{L}\left\{ \begin{Bmatrix} -\bar{n} \\ \bar{n} \end{Bmatrix} \begin{bmatrix} -\bar{n}^t & \bar{n}^t \end{bmatrix} \right\} \tag{1.9}$$

Exercises

1. Use equation 1.5 to calculate the elongation of the truss element given below and compare your answer to the exact value of the elongation.



2. Formulate the stiffness matrix of the three truss elements given below.



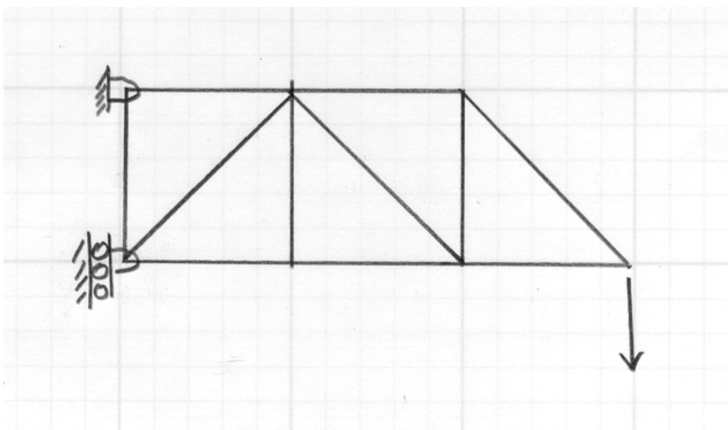
3. Find the values of D_i which make the quadratic form given below stationary.

$$\pi = 8D_1^2 + 4D_2^2 + 10D_3^2 - 4D_1D_2 - 2D_2D_3 - 8D_1 - 7D_3 + 16$$

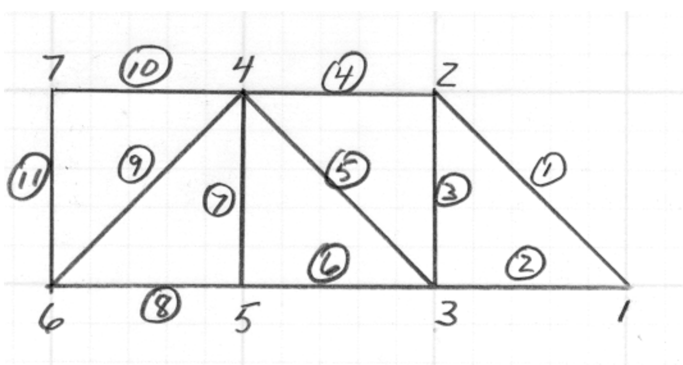
Assembly Procedure

Next we will assemble the strain energy for a whole truss.

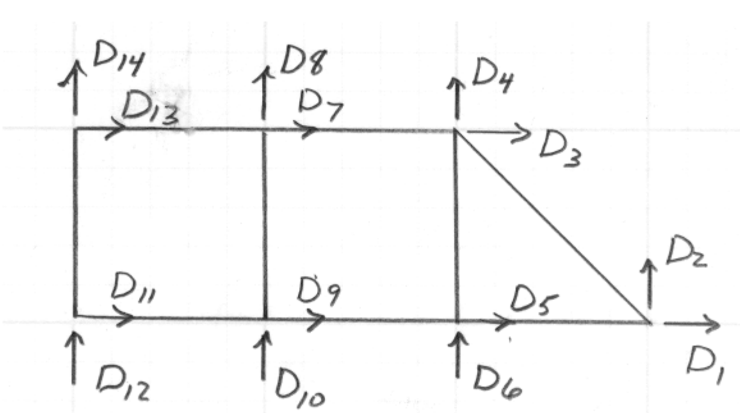
We will illustrate the assemble procedure with a two dimensional truss. The generalization to three dimensions will be obvious. Consider the truss shown below.



Label the elements and nodes as shown below.



Now label the degrees of freedom for each node starting with node 1 as shown below.



Let the stiffness matrix for a truss element be represented by [k]. The strain energy can then be written as:

$$U_e = \frac{1}{2}\overline{d}_e^t[k_e]\overline{d}_e \tag{1.10}$$

where [k_e] is the element stiffness matrix and $\overline{d}_e^t$ is the matrix of displacements for the element with local numbering.

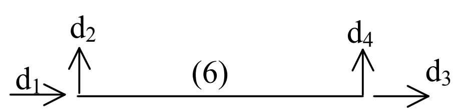
Next we write U_e in terms of the global degrees of freedom as:

$$U_e = \frac{1}{2}\overline{D}^t[K]_e\overline{D} \tag{1.11}$$

where $\overline{D}$ is the matrix of global displacements and $[K]_e$ is the element stiffness matrix expanded to the global numbering system. $[K]_e$ will have zeros everywhere accept in locations that map to the local numbering system for the element. We can represent this map with the table shown below which is constructed for element 6 in the example.

(6)

Local	Global
1	9
2	10
3	5
4	6



We can now assemble the strain energy for the truss which we do as follows:

$$U=U_1+U_2+ \;\ldots\ldots.. \tag{1.12}$$

$$U = \frac{1}{2}\overline{D}^t([K]_1 + [K]_2 + \ldots\ldots..)\overline{D} = \frac{1}{2}\overline{D}^t[K]\overline{D} \tag{1.13}$$

where [K] is called the global stiffness matrix.

Equations of Equilibrium

We are now ready use the principle of Minimum Total Potential Energy to formulate the equations of equilibrium for the truss.

The total potential for the truss can be written as:

$$\pi = \frac{1}{2}\overline{D}^t[K]\overline{D} - \overline{D}^t\overline{F} \tag{1.14}$$

where $\overline{F}$ is a vector of applied forces on the nodes and $\overline{D}$ is the vector of node displacements. It should be noted that the applied forces are numbered in the same manner as the D's and point in the same directions.

Using the principle of Minimum total potential the equilibrium equations can be written as:

$$[K]\overline{D} = \overline{F} \tag{1.15}$$

Boundary Conditions

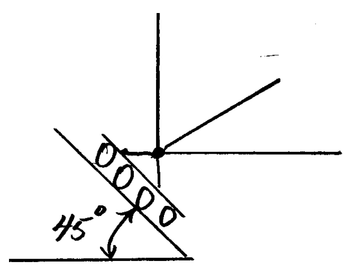
We are now ready to apply boundary conditions to the equilibrium equations.

Referring to the truss example we can list the unknown forces as F_{11}, F_{13} and F_{14}.
The unknown displacements are D_1 through D_{10} and D_{12}.
All other D's and F's are known.

This means we have 14 equations and 14 unknowns to solve for the unknown displacements and forces.

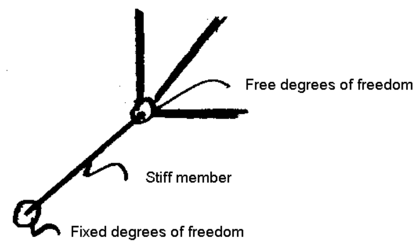
Note that these equations can be solved by first solving equations 1 through 10 and equation 12 for the 11 unknown displacements and then using those displacements in equations 11,13 and 14 to find the three unknown forces. Displacement restraints of the form D_i=const are called single point constraints.

Next change the boundary condition on node 6 of the example truss to that shown below.



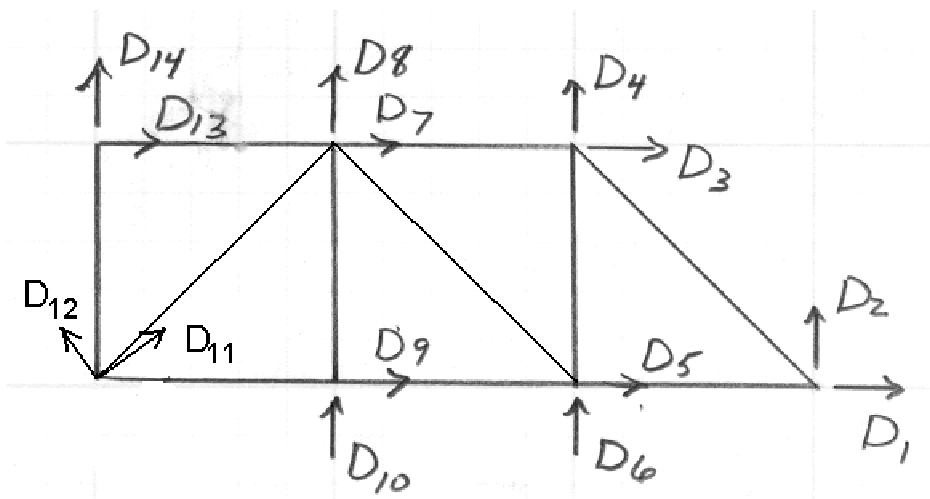
Note that we can not now apply the boundary conditions and solve for the displacements as we did when the displacement boundary conditions aligned with the the directions of the unknown displacements.

We will describe three ways to solve for the displacements of the truss and satisfy the given displacement boundary condition. The first method is to introduce a stiff truss member perpendicular to the incline as shown below and use single point constraints.



Note that the member can be made stiff by increasing it's cross sectional area, making it's length short, increasing it's material's modulus of elasticity or a combination of the above. The member must be stiff enough to constrain the motion of node 6 to a small displacement parallel to the stiff member but not so stiff that it makes the stiffness matrix ill-conditioned. Some trial might be necessary to tune the stiffness. Starting with a stiffness 100 times stiffer than the stiffest member should provide a reasonable solution.

The second method involves referring the displacements of node 6 to a new coordinate system with one axis along the incline and the other perpendicular to the incline and then using a single point constraint to specify that the displacement perpendicular to the incline is zero. This means that the problem needs to be reformulated so that the displacements of node 6 are refered to the new directions as shown below.



We need to refer the total potential energy to the new displacement directions. We could accomplish this in one of two ways. We could refer the strain energy of each bar to the new directions before they are used in the assembly or we could transform the expression for the total potential to the new directions after it has been assembled. We can transform the total potential to the new directions by first writing a transformation matrix [T] between the old and new directions and proceeding as is shown below.

$$\overline{D} = [T]\hat{\overline{D}} \tag{1.16}$$

$$\Pi = 1/2\hat{\overline{D}}^t[T]^t[K][T]\hat{\overline{D}} - \hat{\overline{D}}^t\hat{\overline{F}} \tag{1.17}$$

$$\Pi = 1/2\hat{\overline{D}}^t[\hat{K}]\hat{\overline{D}} - \hat{\overline{D}}^t\hat{\overline{F}} \tag{1.18}$$

Where $$[\hat{K}] = [T]^t[K][T] \tag{1.19}$$

The third method makes use of Lagrange multipliers and constraint equations. In this method we first write a constraint equation relating the displacement variables. For node 6 this can be done as shown below.

$$D_{11}\cos(45) + D_{12}\sin(45) = 0 \tag{1.20}$$

Now to apply the principle of Minimum Total Potential we must find the D's that make Π stationary and satisfy the above constraint equation. This is done by formulating a new functional that includes the constraint equation multiplied by a Lagrange multiplier as shown below.

$$\Pi' = 1/2\overline{D}^t[K]\overline{D} - \overline{D}^t\overline{F} - \lambda(D_{11}\cos(45) + D_{12}\sin(45)) \tag{1.21}$$

This expression can be rewritten as:

$$\Pi' = 1/2\overline{D}^t[K]\overline{D} - \overline{D}^t\overline{F} - \lambda\overline{D}^t\overline{A} \tag{1.22}$$

Where $$\overline{A} = \begin{Bmatrix} 0 \\ 0 \\ \\ \cos(45) \\ \sin(45) \\ \\ \end{Bmatrix} \begin{matrix} \\ \\ \\ 11^{th}\text{ row} \\ 12^{th}\text{ row} \\ \\ \end{matrix} \tag{1.23}$$

The equations to be solved now take the form

$$[K]\overline{D} = \overline{F} + \lambda\overline{A} \tag{1.24}$$

$$\overline{D}^t\overline{A} = \overline{0} \tag{1.25}$$

The above equations will take the form of 15 equations in the 15 unknowns D_1 through D_{12} , F_{13}, F_{14} and λ.

Exercises: