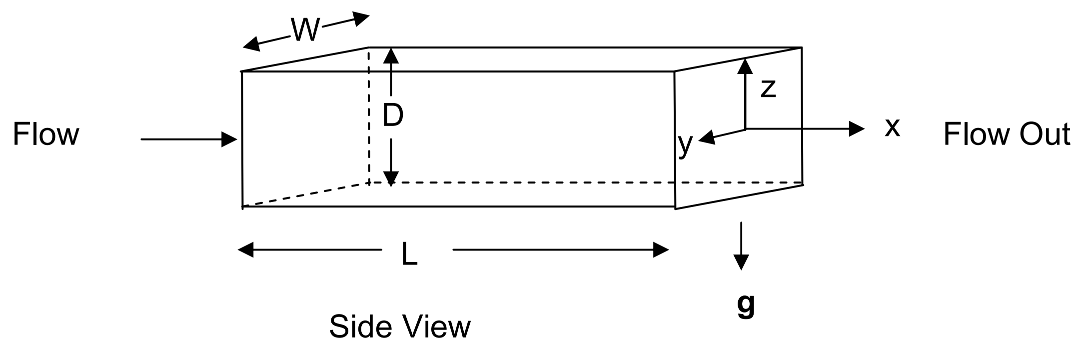
CM 5300, Advanced Transport Phenomena **Spring 2008**
Examination #1 **Mon. 25 Feb., 2008**

Open Book and Open Notes

Problem 1. Steady Laminar Flow Through a Square Duct

A liquid of constant density and viscosity is flowing due to a pressure gradient down a non-converging square duct in laminar flow from left to right, as shown in the figure below. Assume that the flow is steady and gravity is perpendicular to the x direction. Answer the following questions.



a) Circle the choice which is most true for this flow.

$v_x = v_x$ (z only)

$v_x = v_x$ (y, z)

$v_x = v_x$ (y only)

$v_x = v_x$ (x, y,z)

b) Equation of Continuity and of Motion: Choose the correct form of the Equation of Continuity Motion from Appendix B of the text. Simplify these equations to their final form (after eliminating terms). Do not solve this equation.

c) Boundary Conditions: Write all of the boundary conditions for this flow problem.

d) Force on the Wetted Surface of the Square Duct: Suppose that you are able to solve this equation for the x-component of velocity as a function of the coordinate directions relevant to this problem. Write down integral expressions for the force on all of the wetted surfaces of the duct.

Problem 2. Problem 3B.16 of the Text

Use the shell balance approach to derive a balance on x-momentum and show that the governing Equation of Motion for x-momentum is

$$\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y} - \frac{v_o}{\nu} v_x\right) = -\frac{P_o - P_L}{\mu L}$$

where $P_o = p - \rho g(0)$ and $P_L = p - \rho g(L)$ are modified pressures. Solve part a) only. You will find it useful to define a new variable $V_x =$ $v_x/$ y and then use one of the solutions in Appendix C on page 852. Then, calculate the force in the x-direction that the fluid exerts on the plates at $y = 0$ and at $y = B$.

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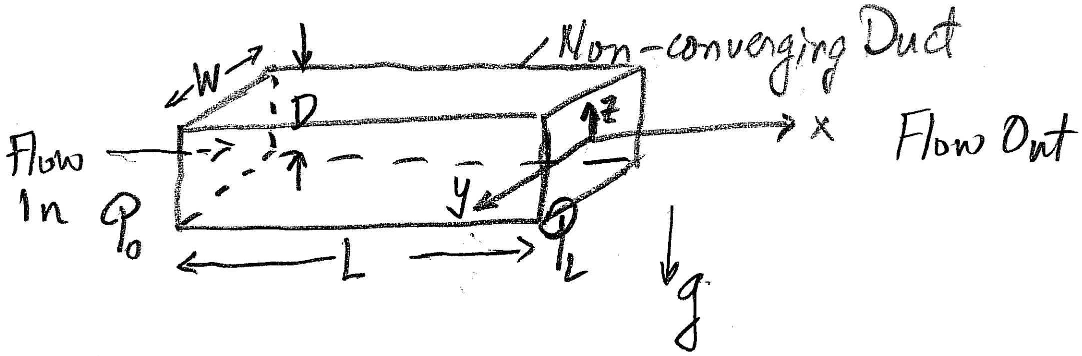
1.



- Steady Laminar Flow,
- $v_y = v_z = 0$
- ρ, μ constant

a) $v_x = v_x(y, z)$

b) Eqns. of Change:

Continuity: Table B.4.

s.s. → steady-state.

$$\cancel{\frac{\partial \rho}{\partial t}}^{0} + \cancel{\frac{\partial}{\partial x}(\rho v_x)}^{0} + \frac{\partial}{\partial y}(\rho \cancel{v_y})^{0} + \frac{\partial}{\partial z}(\rho \cancel{v_z})^{0} = 0$$

$v_x \neq v_x(x)$ $\quad v_y = 0$ $\quad v_z = 0$

all terms $= 0$

x-component of Eqn. of motion: Table B.6

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}}^{\text{S.S.}} + v_x \cancel{\frac{\partial v_x}{\partial x}}^{0} + \cancel{v_y}^{0}\frac{\partial v_x}{\partial y} + \cancel{v_z}^{0}\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}}^{0} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$v_x \neq v_x(x)$ $\qquad v_x \neq v_x(x)$

$$0 = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

let $\mathcal{P} \equiv p - \rho g_x x \rightarrow -\frac{\partial \mathcal{P}}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\rho g_x x)$

$$= -\frac{\partial p}{\partial x} + \rho g_x$$

$$\boxed{0 = -\frac{\partial \mathcal{P}}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]}$$

c) Boundary Conditions:

assume that the width (W) spans $-w \le y \le +w$

depth (D) " $-d \le z \le +d$

BC 1	$y = -w$	$v_x = 0$
BC 2	$y = +w$	$v_x = 0$
BC 3	$z = -d$	$v_x = 0$
BC 4	$z = +d$	$v_x = 0$

d) Force of Fluid on the Wetted Surface of Duct:

$$F(x) = \int_0^L \int_{-d}^{+d} \left(\tau_{yx}\big|_{-w} + \tau_{yx}\big|_{+w} \right) dz\,dx \quad +$$

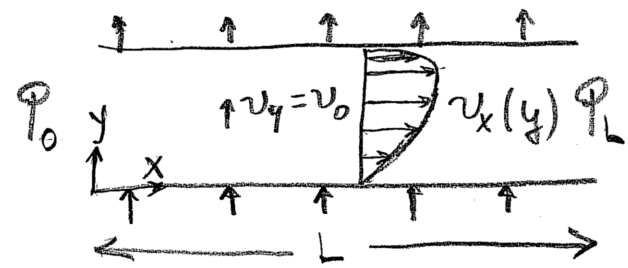
$$\int_0^L \int_{-w}^{+w} \left(\tau_{zx}\big|_{-d} + \tau_{zx}\big|_{+d} \right) dy\,dx$$

where, from Table B.1:

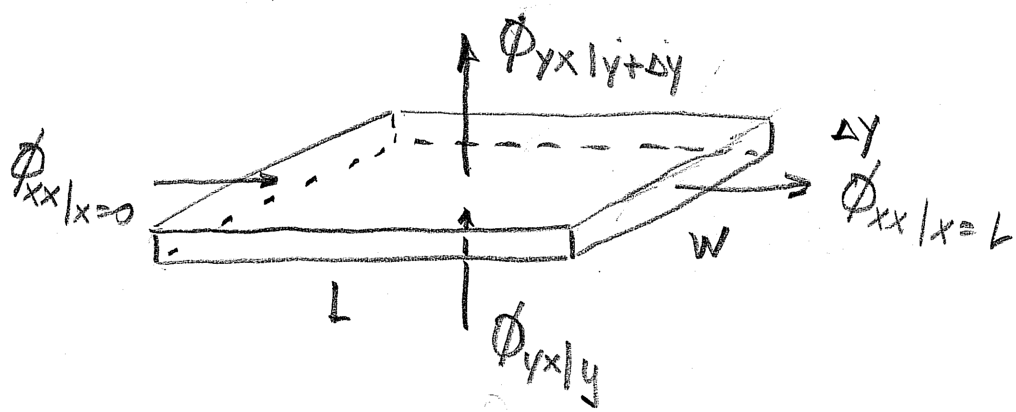
$$\tau_{yx} = -\mu\left[\cancelto{0}{\frac{\partial v_y}{\partial x}} + \frac{\partial v_x}{\partial y}\right] = -\mu\frac{\partial v_x}{\partial y}.$$

$$\tau_{zx} = -\mu\left[\cancelto{0}{\frac{\partial v_z}{\partial x}} + \frac{\partial v_x}{\partial z}\right] = -\mu\frac{\partial v_x}{\partial z}$$

2. Prob. 3B.16



Shell Balance Approach: x-momentum



rate of x-momentum in at $x=0$ $\quad (W\Delta y)(\phi_{xx}+\phi_{xy})|_{x=0}$

" " " out " $x=L$ $\quad (W\Delta y(\phi_{xx}+\phi_{xy})_{x=L}$

" " " in " y $\quad (WL)(\phi_{yx}+\phi_{yy})|_{y=0}$

" " " out " $y+\Delta y$ $\quad (WL)(\phi_{yx}+\phi_{yy})|_{y=L}$

gravity force acting in x-direction $\quad (WL\Delta y)\rho g_x$

$$(WL)((\phi_{yx}|_y - (\phi_{yx}|_{y+\Delta y}) + (W\Delta y)((\phi_{xx}|_{x=0} - (\phi_{xx}|_{x=L}) + (WL\Delta y)\rho g_x = 0$$

$\div$ by $(WL\Delta y)$ $\quad$, $\lim \Delta y \to 0$

$$\frac{\partial \phi_{yx}}{\partial y} - \frac{\phi_{xx}|_{x=0} - \phi_{xx}|_{x=L}}{L} = \rho g_x$$

$$\phi_{yx} = \tau_{yx} + \rho v_y v_x = -\mu \frac{\partial v_x}{\partial y} + \rho v_0 v_x$$

$$\phi_{xx} = p + \tau_{xx} + \rho v_x v_x = p - 2\mu \cancelto{0}{\frac{\partial v_x}{\partial x}} + \rho v_x v_x$$

subst. into eqn. above,

$$\frac{\partial}{\partial y}\left(-\mu \frac{\partial v_x}{\partial y} + \rho v_0 v_x\right) - \frac{(p + \rho v_x v_x)|_{x=0} - (p + \rho v_x v_x)|_{x=L}}{L} = \rho g_x$$

but $v_x|_{x=0} = v_x|_{x=L}$ $\therefore$ $\rho v_x v_x|_{x=0} - \rho v_x v_x|_{x=L} = 0$

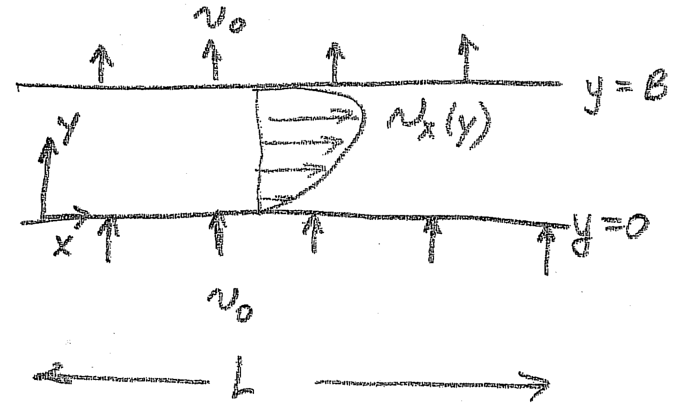
and the gravity term can be combined with p,

$$\frac{\partial}{\partial y}\left(-\mu \frac{\partial v_x}{\partial y} + \rho v_0 v_x\right) - \frac{(p - \rho g_x(0)) - (p - \rho g_x(L))}{L} = 0$$

$$\frac{\partial}{\partial y}\left(-\mu \frac{\partial v_x}{\partial y} + \rho v_0 v_x\right) - \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} = 0$$

$$\boxed{\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y} - \frac{v_0}{\nu} v_x\right) = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}}$$

3B.16



steady flow

ρ, μ constant.

$v_x = v_x(y)$

$v_y = v_0$

$v_z = 0$

a) Show that $v_x = \dfrac{(\mathcal{P}_0 - \mathcal{P}_L)}{\mu L} \dfrac{B^2}{A} \left(\dfrac{y}{B} - \dfrac{e^{Ay/B} - 1}{e^A - 1} \right)$

where $A \equiv \dfrac{B v_0 \rho}{\mu}$

Eqn. of Continuity, Table B.4.

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \frac{\partial}{\partial x}\underbrace{\cancelto{0}{(\rho v_x)}}_{v_x(y) \text{ only}} + \frac{\partial}{\partial y}\underbrace{\cancelto{0}{(\rho v_y)}}_{v_y = \text{constant} = v_0} + \frac{\partial}{\partial z}\cancelto{0}{(\rho v_z)} = 0$$

Eqn of Motion: Table B.6

x-component,

$$\rho \left(\cancelto{0}{\frac{\partial v_x}{\partial t}} + v_x \cancelto{0}{\frac{\partial v_x}{\partial x}} + v_y \frac{\partial v_x}{\partial y} + \cancelto{0}{v_z \frac{\partial v_x}{\partial z}} \right) = -\frac{\partial p}{\partial x} + \mu \left(\cancelto{0}{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancelto{0}{\frac{\partial^2 v_x}{\partial z^2}} \right) + \rho g_x$$

$$\rho v_y \frac{\partial v_x}{\partial y} = -\frac{\partial \mathcal{P}}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2} \qquad ; \mathcal{P} = p - \rho g h$$

$$\frac{\partial^2 v_x}{\partial y^2} - \frac{v_0}{\nu} \frac{\partial v_x}{\partial y} = \frac{1}{\mu} \frac{\partial \mathcal{P}}{\partial x} = \text{Constant}$$

$$\int_{\mathcal{P}_0}^{\mathcal{P}_L} \partial \mathcal{P} = \int_0^L (\text{Constant})\, dx \longrightarrow \mathcal{P}_L - \mathcal{P}_0 = (\text{Constant})\, L$$

$$\text{Constant} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}$$

$$\frac{\partial^2 v_x}{\partial y^2} - \frac{v_0}{\nu}\frac{\partial v_x}{\partial y} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}$$

BC1 $y = 0$ $v_x = 0$

BC2 $y = B$ $v_x = 0$

Solution: let $V_x \equiv \frac{\partial v_x}{\partial y}$, restate eqn. above →

$$\frac{\partial V_x}{\partial y} - \frac{v_0}{\nu} V_x = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}$$

From Appendix C, for $\frac{dy}{dx} + f(x)y = g(x)$.

$$y = \exp\left(-\int f(x)\,dx\right)\left(\int \exp\left(\int f(x)dx\right) g(x)\,dx + C_1\right)$$

$$f(x) = -\frac{v_0}{\nu}, \quad g(x) = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}, \quad y = V_x$$

$$-\int f(x)dx = \int \frac{v_0}{\nu}\,dy = \frac{v_0}{\nu}y$$

$$\int f(x)dx = -\frac{v_0}{\nu}y$$

$$\exp\left(\int f(x)dx\right) = \exp\left(-\frac{v_0}{\nu}y\right)$$

$$\exp\left(-\int f(x)dx\right) = \exp\left(\frac{v_0}{\nu}y\right)$$

$$\int \exp\left(\int f(x)dx\right)g(x)\,dx = \int \exp\left(-\frac{v_0}{\nu}y\right)\left(-\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}\right)dy$$

$$= \frac{\nu}{v_0}\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}\right)\exp\left(-\frac{v_0}{\nu}y\right)$$

$$V_x = \exp\left(\frac{v_0}{\nu}y\right)\left[\frac{\nu}{v_0}\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu L}\right)\exp\left(-\frac{v_0}{\nu}y\right) + C_1\right]$$

$$\frac{\partial v_x}{\partial y} = \frac{\nu}{v_0}\left(\frac{P_0-P_L}{\mu L}\right) + C_1 \exp\left(\frac{v_0}{\nu}y\right) \longrightarrow \text{integrate}$$

$$v_x = \frac{\nu}{v_0}\left(\frac{P_0-P_L}{\mu L}\right)y + C_1\frac{\nu}{v_0}\exp\left(\frac{v_0}{\nu}y\right) + C_2$$

BC1 $\quad y=0 \quad v_x=0 \qquad 0 = C_1\frac{\nu}{v_0} + C_2$

BC2 $\quad y=B \quad v_x=0 \qquad 0 = \frac{\nu}{v_0}\left(\frac{P_0-P_L}{\mu L}\right)B + C_1\frac{\nu}{v_0}\exp\left(\frac{v_0}{\nu}B\right) + C_2$

$\therefore\ C_2 = -C_1\frac{\nu}{v_0}$ from BC1.

BC2 $\rightarrow$ $0 = \frac{\nu}{v_0}\left(\frac{P_0-P_L}{\mu L}\right)B + C_1\frac{\nu}{v_0}\exp\left(\frac{v_0}{\nu}B\right) - C_1\frac{\nu}{v_0}$

$$C_1 = -\frac{B(P_0-P_L)/L\mu}{\exp\left(\frac{v_0}{\nu}B\right)-1}$$

$$C_2 = \frac{\nu}{v_0}\,\frac{B(P_0-P_L)/L\mu}{\exp\left(\frac{v_0}{\nu}B\right)-1}$$

$$v_x = \frac{\nu}{v_0}\left(\frac{P_0-P_L}{L}\right)y - \frac{B(P_0-P_L)/L\mu}{\exp\left(\frac{v_0}{\nu}B\right)-1}\left(\frac{\nu}{v_0}\right)\exp\left(\frac{v_0}{\nu}y\right) + \frac{B(P_0-P_L)/L\mu}{\exp\left(\frac{v_0}{\nu}B\right)-1}\left(\frac{\nu}{v_0}\right)$$

$$= \frac{\nu}{v_0}\left(\frac{P_0-P_L}{\mu L}\right)B\left[\frac{y}{B} - \frac{\exp\left(\frac{v_0}{\nu}y\right)-1}{\exp\left(\frac{v_0}{\nu}B\right)-1}\right]$$

let $A \equiv \frac{B v_0}{\nu} \rightarrow \frac{\nu}{v_0} = \frac{B}{A}$

$$\boxed{v_x = \left(\frac{P_0-P_L}{\mu L}\right)\frac{B^2}{A}\left[\frac{y}{B} - \frac{\exp(Ay/B)-1}{\exp(A)-1}\right]}$$

Force of Fluid in x-direction on plates at $y=0$, $y=B$

$$F_{(x)}\big|_{y=0} = \int_0^W \int_0^L \tau_{yx}\big|_{y=0}\, dx\, dz = WL\left(-\mu \frac{dv_x}{dy}\Big|_{y=0}\right.$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\mathcal{P}_0-\mathcal{P}_L}{\mu L}\right)\frac{B^2}{A}\left[\frac{y}{B} - \frac{\exp(Ay/B)-1}{\exp(A)-1}\right]\right]$$

$$= \left(\frac{\mathcal{P}_0-\mathcal{P}_L}{\mu L}\right)\frac{B^2}{A}\left[\frac{1}{B} - \frac{A\exp(Ay/B)}{B\exp(A)-1}\right]$$

$$\frac{\partial v_x}{\partial y}\Big|_{y=0} = \left(\frac{\mathcal{P}_0-\mathcal{P}_L}{\mu L}\right)\frac{B^2}{A}\left[\frac{1}{B} - \frac{A\exp(0)}{B\exp(A)-1}\right]$$

$$= \left(\frac{\mathcal{P}_0-\mathcal{P}_L}{\mu L}\right)\frac{B^2}{A}\left[\frac{1}{B} - \frac{A}{B\exp(A)-1}\right]$$

$$\boxed{F_{(x)}\big|_{y=0} = -W\left(\frac{\mathcal{P}_0-\mathcal{P}_L}{\mu}\right)\frac{B}{A}\left[1 - \frac{A}{\exp(A)-1}\right]}$$

$$F_{(x)}\big|_{y=B} = \int_0^W \int_0^L \tau_{yx}\big|_{y=B}\, dx\, dz = WL\left(-\mu \frac{dv_x}{dy}\Big|_{y=B}\right)$$

$$\left.\frac{dv_x}{dy}\right|_{y=B} = \left(\frac{P_0 - P_L}{\mu L}\right)\frac{B^2}{A}\left[\frac{1}{B} - \frac{A\exp\left(A\cancel{B}/\cancel{B}\right)}{B\exp(A) - 1}\right]$$

$$\boxed{\left.F_{(x)}\right|_{y=B} = -W\left(\frac{P_0 - P_L}{\mu}\right)\frac{B}{A}\left[1 - \frac{A\exp(A)}{\exp(A) - 1}\right]}$$