\[ \frac{\partial^2 u_y}{\partial y^2} = 0 \left( \nu \alpha \frac{\delta_0}{l_0} \cdot \frac{1}{\delta_0} \right) \quad \Rightarrow \quad \frac{\partial^2 u_y}{\partial x^2} \ll \frac{\partial^2 u_y}{\partial y^2} \]

\[ \nu \frac{\partial^2 u_y}{\partial y^2} = 0 \left( \nu \delta_0 \frac{\delta_0}{l_0^2} \right) \quad \Rightarrow \quad \frac{\partial^2 u_y}{\partial x^2} \ll \frac{\partial^2 u_y}{\partial y^2} \]

From above analyses,

\[ \frac{\partial \delta}{\partial y} = 0 \left( \nu \delta_0 \frac{\delta_0}{l_0^2} \right) \]

\[ \frac{\partial \delta}{\partial x} = 0 \left( \nu \delta_0 / l_0 \right) \]

All terms in \( y \)-component eqn. of motion are smaller by a factor of \( \frac{\delta_0}{l_0} \) compared to \( x \)-component eqn.

For \( Re > 1 \times 10^4 \), \( \frac{\delta_0}{l_0} \leq 10^{-2} \).

Continuity

\[ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \]

\( x \)-motion

\[ \nu_x \frac{\partial u_x}{\partial x} + \nu_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho} \frac{\partial \delta}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2} \]

We assume that the pressure (modified) gradient is known from measurements, or is related to the external velocity \( u_e(x) \) by:

\[ -\frac{1}{\rho} \frac{\partial \delta}{\partial x} = u_e \frac{\partial u_e}{\partial x} \quad \text{for uniform steady flow} \]
Integral Boundary Layer Analysis. (Middleman, pg 401-404)

\[ \rho \left( v_x \frac{\partial u_x}{\partial x} + v_y \frac{\partial u_x}{\partial y} \right) = \mu \frac{\partial^2 u_x}{\partial y^2} \quad \text{x-component} \]

\[ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \quad \text{continuity}. \]

We wish to obtain an approximate solution to \( u_x(y, x) \) and \( \delta(x) \), the boundary layer thickness. We will integrate the x-component eqn from the solid surface \( (y=0) \) to the boundary between the laminar flow near the surface and the turbulent flow, \( \delta(x) \). This will provide a governing eqn. that is averaged over the b.l. Furthermore, we will make assumptions on the form of \( u_x(y, x) \)

First multiply continuity eqn. by \( \rho v_x \) and add this to the x-component eqn.

\[ \rho v_x \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_v}{\partial y} \right) = 0 \]

x-component eqn is now

\[ \rho \left( v_x \frac{\partial u_x}{\partial x} + v_y \frac{\partial u_x}{\partial y} \right) + \rho \left( u_x \frac{\partial u_x}{\partial x} + u_x \frac{\partial u_v}{\partial y} \right) = \mu \frac{\partial^2 u_x}{\partial y^2} \]
\[ \rho 2u_x \frac{\partial u_x}{\partial x} + \rho \nu_y \frac{\partial u_v}{\partial y} + \rho \nu_x \frac{\partial u_x}{\partial y} = \mu \frac{\partial^2 u_x}{\partial y^2} \]
\[ \rho \frac{d^2 u_x}{dx^2} + \rho \frac{\partial}{\partial y} (u_x u_y) = \mu \frac{\partial^2 u_x}{\partial y^2} \]

Now, multiply each term \( dy \) and \( \int_0^\delta \)

\[ \rho \int_0^\delta \frac{d u_x}{dx} \, dy + \rho \int_0^\delta \frac{\partial}{\partial y} (u_x u_y) \, dy = \mu \int_0^\delta \frac{\partial^2 u_x}{\partial y^2} \, dy \]

\[ \int_0^\delta \frac{d u_x}{dx} \, dy + u_x u_y \bigg|_0^\delta = \nu \left[ \frac{d u_x}{dy} \right]_0^\delta \]

Impose boundary conditions:

\( y = 0 \): \( \nu_x = u_y = 0 \)

\( y = \delta \): \( u_x = U \)

\[ \int_0^\delta \frac{d u_x}{dx} \, dy + U u_y \bigg|_{y=\delta} = \nu \left[ \frac{d u_x}{dy} \right]_0^\delta \]

To evaluate the 1st term and also \( u_y \big|_{y=0} \), we must use Leibniz rule of integration because the upper limit of the integral is a function of \( x \)!
Leibniz Rule:
\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, y) \, dy = \int_{a(x)}^{b(x)} \frac{\partial f(x, y)}{\partial x} \, dy + f(x, b(x)) \frac{db}{dx} - f(x, a(x)) \frac{da}{dx}
\]

where \(a(x) = 0\)  \(b(x) = s(x)\)

\[
\int_{0}^{\delta(x)} \frac{\partial v_x}{\partial x} \, dy = \frac{d}{dx} \int_{0}^{\delta(x)} v_x^2 \, dy - v_x^2 (y = \delta(x)) \frac{d\delta(x)}{dx}
\]

\[
= \frac{d}{dx} \int_{0}^{\delta(x)} v_x^2 \, dy - u^2 \frac{d\delta(x)}{dx}
\]

We average the Continuity Eqn across the b.l.
\[
\int_{0}^{\delta(x)} \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) \, dy = \int_{0}^{\delta(x)} \frac{\partial v_x}{\partial x} \, dy + v_y (y = \delta(x)) = 0
\]

\[
\therefore v_y (y = \delta(x)) = - \int_{0}^{\delta(x)} \frac{\partial v_x}{\partial x} \, dy
\]

\[
= - \frac{d}{dx} \int_{0}^{\delta(x)} v_x \, dy + u \frac{d\delta(x)}{dx}
\]

Subst. these results into the x-component eqn. that was integrated across the b.l. we get:
\[
\int_{0}^{\delta(x)} \frac{\partial v_x}{\partial x} \, dy + u \left[ \frac{\partial v_x}{\partial y} \right]_{0}^{\delta(x)} = \nu \left[ \frac{\partial v_x}{\partial y} \right]_{0}^{\delta(x)}
\]

\[
\frac{d}{dx} \int_{0}^{\delta(x)} v_x^2 \, dy - u^2 \frac{d\delta(x)}{dx} - u \frac{d}{dx} \int_{0}^{\delta(x)} v_x \, dy + u^2 \frac{d\delta(x)}{dx} = \nu \left[ \frac{\partial v_x}{\partial y} \right]_{0}^{\delta(x)}
\]
we now evaluate the rhs:

at \( y = S(x) \) we assume that \( \frac{\partial u}{\partial y} = 0 \)

at \( y = 0 \) we define \( \tau_0 = \mu \left[ \frac{\partial u}{\partial y} \right]_{y=0} \)

The avg. x-component eqn. becomes (integral b.d. eqn.).

\[
\frac{d}{dx} \int_0^{S(x)} v_x^2 dy - U \frac{d}{dx} \int_0^{S(x)} v_x dy = -\frac{\tau_0}{\rho}
\]

We have 3 unknowns, \( v_x, \tau_0, \) and \( S(x) \) but only 2 eqns. highlighted above. We now solve for \( S(x) \) by assuming a form for the velocity \( v_x \):

\[
\frac{v_x}{U} = f \left( \frac{y}{S(x)} \right) = f(\eta) \quad \eta = \frac{y}{S(x)}
\]

where \( f \) is yet some unknown function of \( y \) and \( S(x) \).

This assumption states that the \( v_x \) profile has the same "shape" as a function of \( y \) no matter where we are in the \( x \) direction. This seems reasonable!

We now write the integral b.d. eqn. as:

\[
\frac{d}{dx} \rho u^2 \int_0^1 \left( \frac{v_x^2 - Uv_x}{u^2} \right) d\eta = -\tau_0 = \frac{d}{dx} \rho u^2 \int_0^1 (f^2 - f) d\eta
\]