$$\frac{\partial^2 v_y}{\partial y^2} = O\left(v_\infty \frac{\delta_0}{l_0} \cdot \frac{1}{\delta_0^2}\right) \quad \therefore \quad \frac{\partial^2 v_y}{\partial x^2} << \frac{\partial^2 v_y}{\partial y^2}$$

$$\nu \frac{\partial^2 v_y}{\partial y^2} = O\left(v_\infty^2 \frac{\delta_0}{l_0^2}\right) \quad \underline{\underline{\text{also}}}$$

From above analyses,

$$\left.\begin{aligned} \frac{\partial \mathcal{P}}{\partial y} &= O\left(v_\infty^2 \frac{\delta_0}{l_0^2}\right) \\ \frac{\partial \mathcal{P}}{\partial x} &= O\left(v_\infty^2 / l_0\right) \end{aligned}\right\} \quad \frac{\partial \mathcal{P}}{\partial y} << \frac{\partial \mathcal{P}}{\partial x}$$

All terms in y-component eqn. of motion are smaller by a factor of $\frac{\delta_0}{l_0}$ compared to x-component eqn!

For $Re > 10^4$, $\frac{\delta_0}{l_0} \le 10^{-2}$!

continuity $\qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

x-motion $\qquad v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}$

We assume that the pressure (modified) gradient is known from measurements, or is related to the external velocity, $v_e(x)$ by.

$$-\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial x} = v_e \frac{d v_e}{dx} \equiv \text{for uniform steady flow} = 0$$

Integral Boundary Layer Analysis. (Middleman, pg 401-404)

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = \mu \frac{\partial^2 v_x}{\partial y.} \qquad x\text{-component}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{continuity.}$$

We wish to obtain an approximate solution to $v_x(y,x)$ and $\delta(x)$, the boundary layer thickness. We will integrate the x-component eqn from the solid surface $(y=0)$ to the boundary between the laminar flow near the surface and the turbulent flow, $\delta(x)$! This will provide a governing eqn. that is averaged over the b.l. Furthermore, we will make assumptions on the form of $v_x(y,x)$

First multiply Continuity Eqn. by ρv_x and add this to the x-component eqn.

$$\rho v_x\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = 0$$

x-component eqn is now.

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) + \rho\left(v_x \frac{\partial v_x}{\partial x} + v_x \frac{\partial v_y}{\partial y}\right) = \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$\rho\, 2 v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + \rho v_x \frac{\partial v_y}{\partial y} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

note: $\rho\, 2 v_x \dfrac{\partial v_x}{\partial x} = \rho \dfrac{\partial v_x^2}{\partial x}$

$$\rho\, v_y \frac{\partial v_x}{\partial y} + \rho\, v_x \frac{\partial v_y}{\partial y} = \rho \frac{\partial}{\partial y}(v_x v_y)$$

now this modified x-component eqn. is

$$\rho \frac{\partial v_x^2}{\partial x} + \rho \frac{\partial}{\partial y}(v_x v_y) = \mu \frac{\partial^2 v_x}{\partial y^2}$$

now. multiply each term dy and $\int_0^{\delta(x)}$

$$\rho \int_0^{\delta(x)} \frac{d v_x^2}{dx}\, dy + \rho \int_0^{\delta(x)} \frac{\partial}{\cancel{\partial y}}(v_x v_y)\, \cancel{dy} = \int_0^{\delta(x)} \mu \frac{\partial}{\cancel{\partial y}}\left(\frac{\partial v_x}{\partial y}\right) \cancel{dy}$$

$$\int_0^{\delta(x)} \frac{d v_x^2}{dx}\, dy + \left[v_x v_y\right]_0^{\delta(x)} = \nu \left[\frac{\partial v_x}{\partial y}\right]_0^{\delta(x)}$$

impose boundary conditions

$y=0$. $v_x = v_y = 0$.

$y = \delta(x)$ $v_x = U$

$$\int_0^{\delta(x)} \frac{d v_x^2}{dx}\, dy + U v_{y(y=\delta)} = \nu \left[\frac{\partial v_x}{\partial y}\right]_0^{\delta(x)}$$

To evaluate the 1st term and also $v_{y(y=0)}$, we must use Leibniz rule of integration because the upper limit of the integral is a function of x!

Leibniz Rule.

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\,dy + f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx}$$

where $a(x)=0$ $\quad b(x)=\delta(x)$

$$\therefore \int_0^{\delta(x)} \frac{\partial v_x^2}{\partial x}\,dy = \frac{d}{dx}\int_0^{\delta(x)} v_x^2\,dy - v_{x\,(y=\delta(x))}^2 \frac{d\delta(x)}{dx}$$

$$= \frac{d}{dx}\int_0^{\delta(x)} v_x^2\,dy - U^2\frac{d\delta(x)}{dx}$$

We average the Continuity Eqn across the b.l.

$$\int_0^{\delta(x)} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)dy = \int_0^{\delta(x)} \frac{\partial v_x}{\partial x}\,dy + v_{y\,(y=\delta(x))} = 0$$

$$\therefore v_{y\,(y=\delta(x))} = -\int_0^{\delta(x)} \frac{\partial v_x}{\partial x}\,dy$$

$$= -\frac{d}{dx}\int_0^{\delta(x)} v_x\,dy + U\frac{d\delta(x)}{dx}$$

Subst. these results into the x-component eqn. that was integrated across the b.l. we get.

$$\int_0^{\delta(x)} \frac{d v_x^2}{dx}\,dy + U v_{y\,(y=\delta(x))} = \nu\left[\frac{\partial v_x}{\partial y}\right]_0^{\delta(x)}$$

$$\frac{d}{dx}\int_0^{\delta(x)} v_x^2\,dy - U^2\frac{d\delta(x)}{dx} - U\frac{d}{dx}\int_0^{\delta(x)} v_x\,dy + U^2\frac{d\delta(x)}{dx} = \nu\left[\frac{\partial v_x}{\partial y}\right]_0^{\delta(x)}$$

we now evaluate the rhs:

at $y=\delta(x)$ we assume that $\dfrac{\partial v_x}{\partial y}=0$

at $y=0$ we define $\boxed{\tau_0=\mu\left[\dfrac{\partial v_x}{\partial y}\right]_{y=0}}$

The avg. x-component eqn becomes. (integral b.l. eqn.).

$$\boxed{\frac{d}{dx}\int_0^{\delta(x)} v_x^2\,dy - U\frac{d}{dx}\int_0^{\delta(x)} v_x\,dy = -\frac{\tau_0}{\rho}}$$

We have 3 unknowns, v_x, τ_0, and $\delta(x)$ but only 2 eqns. highlighted above. We now solve for $\delta(x)$ by assuming a form for the velocity $v_x\left(\frac{y}{\delta(x)}\right)$

$$\frac{v_x}{U}=f\left(\frac{y}{\delta(x)}\right)=f(\eta) \qquad \eta\equiv\frac{y}{\delta(x)}$$

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where f is yet some unknown function of y and $\delta(x)$. This assumption states that the v_x profile has the same "shape" as a function of y no matter where we are in the x direction. This seems reasonable!

we now write the integral b.l. eqn as.

$$\frac{d}{dx}\rho U^2\delta\int_0^1\left(\frac{v_x^2-Uv_x}{U^2}\right)d\eta=-\tau_0=\frac{d}{dx}\rho U^2\delta\int_0^1(f^2-f)\,d\eta$$