The Navier-Stokes Eqns for v_r, v_θ are.

v_r:

$$\rho\left(\cancelto{0}{\frac{\partial v_r}{\partial t}} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \cancelto{0}{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}} - \frac{v_\theta^2 + \cancelto{0}{v_\phi^2}}{r}\right) =$$

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cancelto{0}{\frac{\partial^2 v_r}{\partial\phi^2}}\right] + \rho g_r$$

v_θ:

$$\rho\left(\cancelto{0}{\frac{\partial v_\theta}{\partial t}} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \cancelto{0}{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}} + \frac{v_r v_\theta - \cancelto{0}{v_\phi^2\cot\theta}}{r}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\cancelto{0}{\frac{\partial^2 v_\theta}{\partial\phi^2}}\right.$$

$$\left. + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\cancelto{0}{\frac{\partial v_\phi}{\partial\phi}}\right] + \rho g_\theta$$

These eqns. can be made dimensionless

$$s = \frac{r}{R}, \quad \tilde{v}_r = \frac{v_r}{U}, \quad \tilde{v}_\theta = \frac{v_\theta}{U}, \quad \tilde{p} = \frac{p - \rho g z}{\mu U/R}$$

r-component

$$\frac{\rho U R}{\mu}\left[\tilde{v}_r\frac{\partial\tilde{v}_r}{\partial s} + \frac{\tilde{v}_\theta}{s}\frac{\partial\tilde{v}_r}{\partial\theta} - \frac{\tilde{v}_\theta^2}{s}\right] = -\frac{\partial\tilde{p}}{\partial s} + \left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}(s^2\tilde{v}_r) + \frac{1}{s^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{v}_r}{\partial\theta}\right)\right]$$

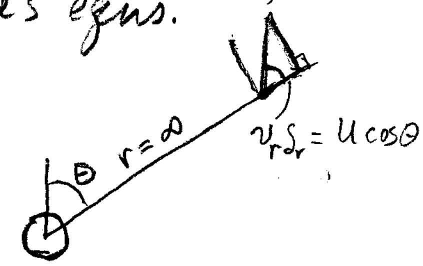
θ-component

$$\frac{\rho U R}{\mu}\left[\tilde{v}_r\frac{\partial\tilde{v}_\theta}{\partial s} + \frac{\tilde{v}_\theta}{s}\frac{\partial\tilde{v}_\theta}{\partial\theta} + \frac{\tilde{v}_r v_\theta}{s}\right] = -\frac{1}{s}\frac{\partial\tilde{p}}{\partial\theta} + \left[\frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\tilde{v}_\theta}{\partial s}\right) + \frac{1}{s^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\tilde{v}_\theta\sin\theta)\right) + \frac{2}{s^2}\frac{\partial\tilde{v}_r}{\partial\theta}\right]$$

↑ Reynold's Number

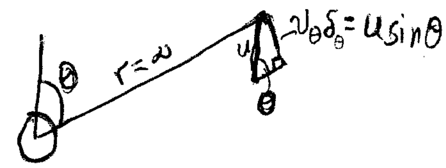
For very small Reynold's No.s we may neglect the lhs of each eqn. The remaining eqn. are the "Stoke's" eqns.

Boundary Conditions:



$r = R$	$v_r = 0$
$r = R$	$v_\theta = 0$
$r = \infty$	$v_r = U\cos\theta$
$r = \infty$	$v_\theta = -U\sin\theta$

the component of U in the r-direction is



<u>Solution:</u>

Often the value at the boundary is reflected in the solution.
Therefore, we try: "separation of variables" method,

$$\tilde{v}_r = F(s)\cos\theta \qquad \tilde{v}_\theta = G(s)\sin\theta$$

When this trial soln. is subst. into the Continuity Eqn.

Show that,

$$\boxed{\left[F(s) + \frac{s}{2}\frac{\partial F(s)}{\partial s}\right] + G(s) = 0}$$

relationship between $G(s)$ and $F(s)$, $\frac{\partial F(s)}{\partial s}$

When the trial solutions are substituted into the r- and θ-component ~~Navier~~-Stokes Eqns., we express these eqns. in terms of $F(s)$, $G(s)$, and $\tilde{p}(s,\theta)$. We eliminate the $\tilde{p}(s,\theta)$ term by performing the following steps on the Stokes Eqns.

<u>r-component eqn:</u> $\frac{\partial}{\partial\theta}$ each term, introduce $\tilde{v}_r = F(s)\cos\theta$
$\tilde{v}_\theta = G(s)\sin\theta$

$$0 = -\frac{\partial\tilde{p}}{\partial s} + \left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}(s^2\tilde{v}_r) + \frac{1}{s^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{v}_r}{\partial\theta}\right)\right]$$

$$\frac{\partial}{\partial\theta}(0) = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}\left(s^2\frac{\partial\tilde{v}_r}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\cdot\frac{1}{s^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{v}_r}{\partial\theta}\right)\right]$$

$$0 = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}\left(s^2\frac{\partial}{\partial\theta}F(s)\cos\theta\right) + \frac{\partial}{\partial\theta}\cdot\frac{1}{s^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(F(s)\cos\theta)\right)\right]$$

$$0 = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}(s^2F(s)(-\sin\theta)) + \frac{\partial}{\partial\theta}\cdot\frac{1}{s^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F(s)(-\sin\theta))\right]$$

$$0 = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}(s^2F(s)(-\sin\theta)) + \frac{\partial}{\partial\theta}\cdot\frac{1}{s^2\cancel{\sin\theta}}\cdot(-2F(s)\cancel{\sin\theta}\cos\theta)\right]$$

$$0 = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}(s^2F(s)(-\sin\theta)) + \frac{(-2)F(s)}{s^2}(-\sin\theta)\right]$$

$$0 = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{(-1)}{s^2}\frac{\partial^2}{\partial s^2}(s^2F(s)) + \frac{2F(s)}{s^2}\right]\sin\theta \qquad *$$

<u>Θ-component eqn.</u> $\cdot s$, introduce $\tilde{v}_r = F(s)\cos\Theta$, $\tilde{v}_\Theta = G(s)\sin\Theta$, $\frac{\partial}{\partial s}$

$$s\,(0) = -s\cdot\frac{1}{s}\frac{\partial\tilde{p}}{\partial\theta} + s\cdot\left[\frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\tilde{v}_\theta}{\partial s}\right) + \frac{s}{s^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\tilde{v}_\theta\sin\theta)\right) + \frac{2s}{s^2}\frac{\partial\tilde{v}_r}{\partial\theta}\right]$$

$$\frac{\partial}{\partial s}(0) = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{\partial}{\partial s}\cdot\frac{1}{s}\frac{\partial}{\partial s}\left(s^2\frac{\partial\tilde{v}_\theta}{\partial s}\right) + \frac{\partial}{\partial s}\cdot\frac{1}{s}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\tilde{v}_\theta\sin\theta)\right) + \frac{\partial}{\partial s}\cdot\frac{2}{s}\frac{\partial\tilde{v}_r}{\partial\theta}\right]$$

$$0 = -\frac{\partial\tilde{p}}{\partial s\partial\theta} + \left[\frac{\partial}{\partial s}\cdot\frac{1}{s}\frac{\partial}{\partial s}\left(s^2\frac{\partial}{\partial s}(G(s)\sin\theta)\right) + \frac{\partial}{\partial s}\cdot\frac{1}{s}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(G(s)\sin^2\theta)\right) + \frac{\partial}{\partial s}\cdot\frac{2}{s}\frac{\partial}{\partial\theta}(F(s)\cos\theta)\right] \div$$

We now subtract θ-component eqn. from r-component eqn, eliminating $\tilde{p}$ term!

$$0 = \left[\frac{(-1)}{s^2}\frac{\partial^2}{\partial s^2}(s^2F(s)) + \frac{2F(s)}{s^2}\right]\sin\theta$$

$$\left[\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s}\left(s^2\frac{\partial}{\partial s}(G(s)\sin\theta)\right) + \frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(G(s)\sin^2\theta\right) + \frac{\partial}{\partial s}\frac{2}{s}\frac{\partial}{\partial\theta}(F(s)\cos\theta)\right]$$

see next

now subst. $G(s) = -\left[F(s) + \frac{s}{2}\frac{\partial F(s)}{\partial s}\right]$ from continuity

everywhere $G(s)$ appears.

After much algebra - - -

2 / 2 _ 1

(45h)

We obtain the following eqn. for $F(s)$

$$s^4\frac{d^4F(s)}{ds^4} + 8s^3\frac{d^3F(s)}{ds^3} + 8s^2\frac{d^2F(s)}{ds^2} - 8s\frac{dF(s)}{ds} = 0$$

Soln: assume $F(s) = s^n$

Solve for the roots of the subsequent equation in terms of n!

$n = -3, -1, 0, 2$ Show this.

~~Show that the~~

the general soln is:

$$F(s) = a s^{-3} + b s^{-1} + c s^{0} + d s^{2}$$

BC 1	$s=1$	$F=0$	($\sim_r$ BCs)
BC 2	$s=\infty$	$F=1$	
BC 3	$s=1$	$G=0$	$\xrightarrow[\text{continuity eqn.}]{\text{(using}} \frac{dF}{ds}=0$
BC4	$s\to\infty$	$G=-1$	$\xrightarrow[\text{result)}]{} \frac{dF}{ds}=0$

after applying these 4 BCs.

$$F(s) = 1 - \frac{3}{2s} + \frac{1}{2s^3}$$

$$G(s) = -1 + \frac{3}{4s} + \frac{1}{4s^3}$$

Determine $\tilde{P}$, subst solns. for $F(s)$ and $G(s)$ into the θ-component "Stokes Eqn." and integrate $d\theta$.

$$\tilde{P} = \tilde{P}_\infty - \frac{3}{2s^2}\cos\theta$$

Wed 2/8/06

Now we wish to calculate the force exerted by the flow on the falling sphere. There are two components acting on the surface.

normal stress component, $\Pi_{rr} = P_r + \underbrace{\tau_{rr}}_{\to 0}$ at a fluid-solid surface for newtonian fluids.

shear stress, $\Pi_{r\theta}$, See pg 59-61