45m

Stream Function: Section 4.2. also Middleman text. p366-371

The Stream Function and solutions in terms of the Stream Function are very useful in solving 2-dimensional steady flows of Newtonian fluids. For 2-dim. flows, we have 3 dependent variables (2 velocity components + pressure) to solve for. By using the Stream Function, we are able to reduce this to just one dependent variable!

The Stream Function also has a geometric interpretation, as we will demonstrate next.

In Cartesian coordinates and for 2-dim. flow, we have 2 velocity components which can be expressed
* in terms of the Stream Function,

$\boxed{\text{Definition of } \psi}$ $v_x = \frac{\partial \psi}{\partial y}(-)$ $\qquad v_y = \frac{\partial \psi}{\partial x}(+)$

where $\psi \equiv$ stream function. $\psi = \psi(x,y)$ varies in x and y directions for 2-dim. flow. A differential change in ψ can be related to dx and dy,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

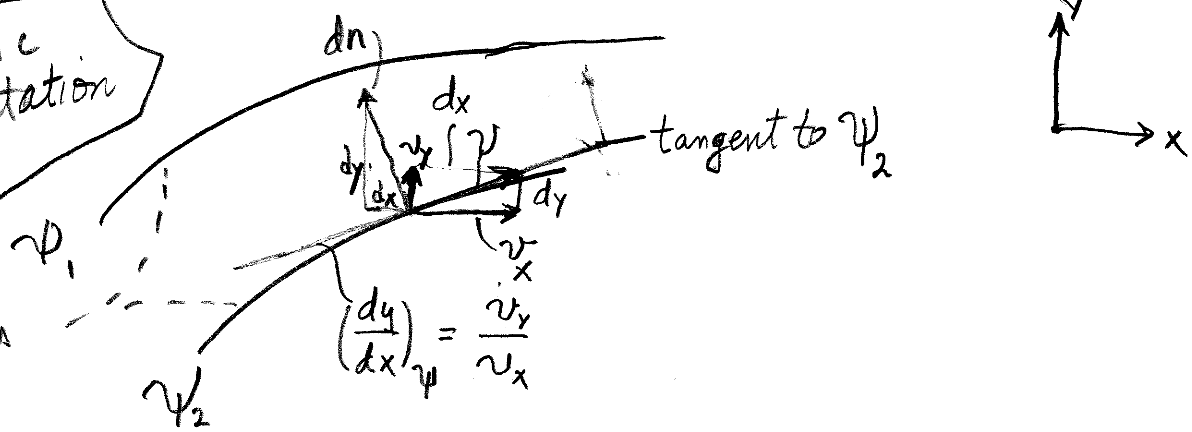
$$= v_y dx - v_x dy$$ (using definitions of ψ from above.)

Along a line of constant ψ, $d\psi = 0$

$$\left[\frac{dy}{dx}\right]_\psi = \frac{v_y}{v_x}$$

In other words, the path that would be taken by a fluid element through space $[dy/dx]_\psi$ at a constant value of ψ ($d\psi = 0$) would be alligned with the fluid velocity (v_y/v_x) components!

Geometric Interpretation of ψ



$$\left(\frac{dy}{dx}\right)_\psi = \frac{v_y}{v_x}$$

streamlines

Thus, the flow path within the fluid are along paths of constant ψ, thus the Stream Function is valuable for

$\boxed{\text{flow visualization.}}$

Another useful property of ψ is its relationship to the mass flow rate.

$$\dot{m} = \int_{\psi_1}^{\psi_2} \rho v \, dn$$

where

$\dot{m} = \dfrac{\text{mass}}{\text{time} \cdot \text{unit width in z-direction}}$

ρ = fluid density $\left(\dfrac{\text{mass}}{\text{volume}}\right)$

v = velocity vector (magnitude only)
$= v_x^2 + v_y^2$

dn = a differential distance $\perp$ to a stream line (const. ψ).

To complete this analysis, we need to relate $d\psi$ to dn and to v.

We note that $\dfrac{dx}{dn} = +\dfrac{v_y}{v} \qquad \dfrac{dy}{dn} = \dfrac{v_x}{v}$

$$\frac{d\psi}{dn} = \left(\frac{\partial \psi}{\partial x}\right)\frac{dx}{dn} + \left(\frac{\partial \psi}{\partial y}\right)\frac{dy}{dn}$$

$$= +v_y \frac{dx}{dn} + v_x \frac{dy}{dn}$$

$$= +v_y\left(+\frac{v_y}{v}\right) + v_x\left(\frac{v_x}{v}\right)$$

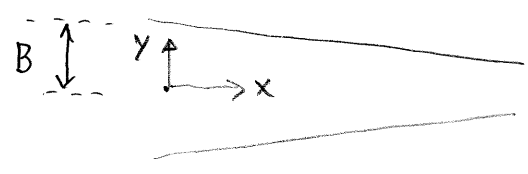
$$= \frac{v_y^2 + v_x^2}{v} = \frac{v^2}{v} = \boxed{v = \frac{\partial \psi}{\partial n}}$$

$$\dot{m} = \int_{\psi_1}^{\psi_2} \rho v \, dn = \int_{\psi_1}^{\psi_2} \rho \frac{\partial \psi}{\cancel{\partial n}} \cancel{dn} = \int_{\psi_1}^{\psi_2} \rho \, d\psi = \boxed{\rho(\psi_2 - \psi_1)}$$

Thus, the mass flow rate between streamlines is proportional to the difference between them. Thus, if ψ is displayed at equal $\Delta\psi$ between them, regions of the greatest mass flowrate will have streamlines closest together.

Example: Laminar Flow Between Parallel Plates:

The general 2-dim. flow case is when the plates are not exactly parallel.



$$v_x = v_x(x,y)$$

$$v_y = v_y(x,y).$$

The Continuity Eqn and Navier–Stokes Eqns. are.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

3 eqns.
3 unknowns.
$\underline{v_x, v_y, P}$

Continuity.

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \quad \text{x-component}$$

$$\rho\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \quad \text{y-component}$$

In terms of ψ. the Continuity Eqn. becomes.

$$-\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

which is satisfied for any $\psi(x,y)$!

Thus we loose 1 eqn, but we have only 2 unknowns, $\underline{\psi, p}$.

We eliminate p by,

subst. definition of ψ into x- and y-component eqns.

$\frac{\partial}{\partial y}$ x-comp. eqn.

$\frac{\partial}{\partial x}$ y-component eqn.

subt. y-comp. eqn. from x-comp. eqn., $\frac{\partial^2 p}{\partial x \partial y}$ terms cancel.

$$\boxed{\frac{\partial \psi}{\partial x}\frac{\partial(\nabla^2\psi)}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial(\nabla^2\psi)}{\partial x} = \nu \nabla^4 \psi}$$

$$\nabla^2 \psi \equiv \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$\nabla^4 \psi \equiv \frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^2}$$

see Table 4. for ψ eqns in other coord. systems.