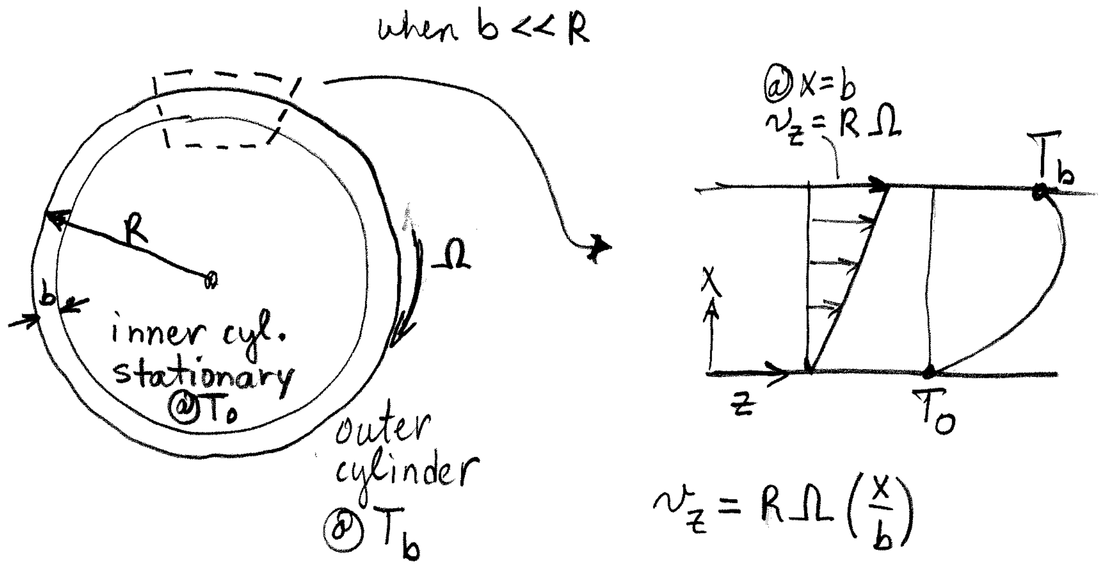
10.4 Heat Conduction With a Viscous Heat Source:



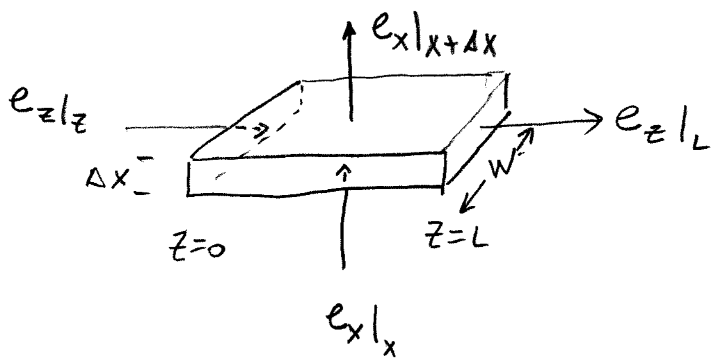
Assumptions;

$v_z = v_z(x)$ only.

$T = T(x)$ only.

no transport in y-direction

Energy Shell Balance;



rate of combined energy in at x	$WL\, e_x\vert_x$
" " " " out " x+Δx	$WL\, e_x\vert_{x+\Delta x}$
" " " " in at z=0	$\Delta x W\, e_z\vert_{z=0}$
" " " " out at z=L	$\Delta x W\, e_z\vert_{z=L}$

$$WL\, e_x|_x - WL\, e_x|_{x+\Delta x} + \Delta x W e_z|_z - \Delta x W e_z|_{z+\Delta z} = 0$$

$\div$ by $\Delta x W L$

$$\frac{e_x|_x - e|_{x+\Delta x}}{\Delta x} + \frac{e_z|_0 - e_z|_L}{L} = 0$$

$$e_x = \delta_x \cdot e = \delta_x \cdot \left[\left(\tfrac{1}{2}\rho v^2 + \rho \hat{H}\right) v + [\tau \cdot v] + q\right]$$

$$= \left(\tfrac{1}{2}\rho v^2 + \rho \hat{H}\right) \cancel{v_x}^{\,0} + [\tau \cdot v]_x + q_x$$

~~where~~ $\cancel{v^2 = (v_x^2 + v_y^2 + v_z^2)^{1/2} = v_z}$

$$[\tau \cdot v]_x = \tau_{xx} \cancel{v_x}^{\,0} + \tau_{xy} \cancel{v_y}^{\,0} + \tau_{xz} v_z = \tau_{xz} v_z$$

$q_x = -k \dfrac{\partial T}{\partial x}$ from Fourier's Law.

$$\boxed{e_x = \tau_{xz} v_z + q_x = -\mu v_z \frac{\partial v_z}{\partial x} - k \frac{\partial T}{\partial x}}$$

$$e_z = \delta_z \cdot e = \left(\tfrac{1}{2}\rho v^2 + \rho \hat{H}\right) v_z + [\tau \cdot v]_z + q_z$$

but $v_z|_{z=0} = v_z|_{z=L}$

$$[\tau \cdot v]_z = \tau_{zx} \cancel{v_x}^{\,0} + \tau_{zy} \cancel{v_y}^{\,0} + \tau_{zz} v_z = -2 \cancel{\left(\frac{\partial v_z}{\partial z}\right)}^{\,0} v_z$$

$$q_z = -k \cancel{\frac{\partial T}{\partial z}}^{\,0}$$

$$\therefore \frac{e_z|_0 - e_z|_L}{L} = 0$$

Shell Energy Balance Simplifies to.

$$\frac{de_x}{dx} = 0$$

integrating: $e_x = C_1$

or $-k\frac{dT}{dx} - \mu v_z \frac{\partial v_z}{dx} = C_1$

but $\frac{dv_z}{dx} = \frac{R\Omega}{b}$.

$$-k\frac{dT}{dx} - \mu x\left(\frac{R\Omega}{b}\right)^2 = C_1$$

note: $\mu\left(\frac{R\Omega}{b}\right)^2$ is rate of viscous heat production/(volume).

Integrating,

$$T = -\left(\frac{\mu}{k}\right)\left(\frac{R\Omega}{b}\right)^2\frac{x^2}{2} - \frac{C_1}{k}x + C_2.$$

BC1 $\quad x = 0 \quad T = T_0$

BC2 $\quad x = b \quad T = T_b$

BC1 $\quad T_0 = -0 - 0 + C_2 \Rightarrow C_2 = T_0$

BC2 $\quad T_b = -\frac{\mu}{k}\left(\frac{R\Omega}{b}\right)^2\frac{b^2}{2} - \frac{C_1}{k}b + T_0$

$$\therefore C_1 = -\frac{(T_b - T_0)k}{b} - \frac{\mu(R\Omega)^2}{2b}$$

$$T = -\frac{\mu}{k}\left(\frac{R\Omega}{b}\right)^2\frac{x^2}{2} - \left[-\frac{(T_b - T_0)}{b} - \frac{\mu(R\Omega)^2}{2kb}\right]x + T_0$$

$$T - T_0 = \frac{\mu(R\Omega)^2(T_b - T_0)}{2k(T_b - T_0)}\left[\frac{x}{b} - \left(\frac{x}{b}\right)^2\right] + (T_b - T_0)\frac{x}{b}$$

$$\frac{T-T_o}{T_b-T_o} = \frac{1}{2}\,\frac{\mu(R\Omega)^2}{k(T_b-T_o)}\,\frac{x}{b}\left(1-\frac{x}{b}\right)+\frac{x}{b}.$$

$$\boxed{= \frac{1}{2}\,Br\,\frac{x}{b}\left(1-\frac{x}{b}\right)+\frac{x}{b}}$$

$Br \equiv$ Brinkman No.

$$= \frac{\mu(R\Omega)^2}{k(T_b-T_o)}$$

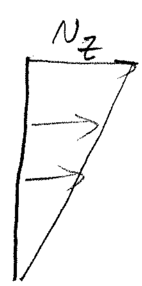
$\mu(R\Omega)^2 \leftarrow$ viscous heat generation rate

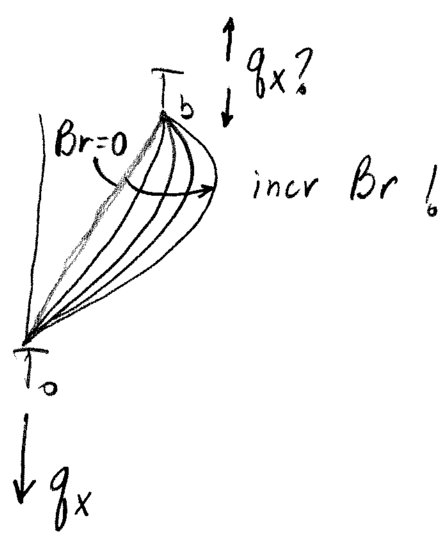
$k(T_b-T_o) \leftarrow$ heat conduction rate.

When $Br=0$ for $\Omega=0$

$$\frac{T-T_o}{T_b-T_o} = \frac{x}{b}$$

, a linear temperature profile as expected from Fourier's Law!





Q:

For what Br does $q_x=0$ @ $x=b$?

$$q_x\big|_b = -k\frac{\partial T}{dx}\bigg|_b = 0 \quad \text{or} \quad \frac{dT}{dx}\bigg|_b = 0$$

$$\frac{dT}{dx} = \frac{1}{2}(T_b - T_o)\,Br\left(\frac{1}{b} - 2\frac{x}{b^2}\right) + \frac{1}{b}(T_b - T_o)$$

$$\frac{dT}{dx}\bigg|_{x=b} = \frac{1}{2}(T_b - T_o)\,Br\left(\frac{1}{b} - \frac{2}{b}\right) + (T_b - T_o)\frac{1}{b}$$

$$= \frac{(T_b - T_o)}{b}\left(1 - \frac{Br}{2}\right)$$

$$\frac{(T_b - T_o)}{b}\left(1 - \frac{Br}{2}\right) = 0. \Rightarrow \boxed{Br = 2}$$

Q2. What is Br for a lubricating oil in an automobile engine? $T_o = 400\ K$

*

modified for $T_b = T_o$

$$\Omega = 3000\ rpm \cdot 2\pi = 314.2\ s^{-1}$$

$$\mu = 100\ mPa \cdot S \qquad Pa = N/m^2 = \frac{kg}{m \cdot s^2}$$

$$k = 0.3\ W/m \cdot {}^\circ K \Rightarrow kg \cdot m/(s^3 \cdot {}^\circ K)$$

$$R = 5\ cm = .05\ m$$

$$Br \equiv \frac{\mu (R\Omega)^2}{k T_o} = \frac{\left(\frac{100}{10^3}\right) kg/(m \cdot s)\ (.05\ m)^2 (314.2\ s^{-1})^2}{(.3\ kg \cdot m/(s^3 \cdot K))(400\ K)} \boxed{= 0.21}$$

For small $\Delta T = T_b - T_0$ and small Br, this solution approaches the exact solution. (k & μ constant).

But how do we handle the general case when we can not assume that k and μ are constant? In this case we can not specify $v_z(x)$ a-priori, but must solve simultaneously for T and $v_z(x)$!

The governing equations now are.

$$-k(T)\frac{\partial T}{\partial x} - \mu(T)\, v_z \frac{\partial v_z}{\partial x} = C_1 \qquad \underline{\text{Energy}}$$

$x=0,\ T=T_0 \qquad x=b,\ T=T_b$

$$\frac{\partial}{\partial x}\left(\mu(T)\frac{\partial v_z}{\partial x}\right) = 0 \qquad \text{Eqn. of Motion}$$

(from B.5).

$x=0,\ v_z=0$
$x=b,\ v_z=0$

Clearly, $T(x)$ and $v(x)$ must be solved for simultaneously, since $T(x)$ depends upon v_z and $v_z(x)$ depends on T (through $\mu(T)$)! A computer-aided numerical soln. is needed!

Mon. 2/27/06