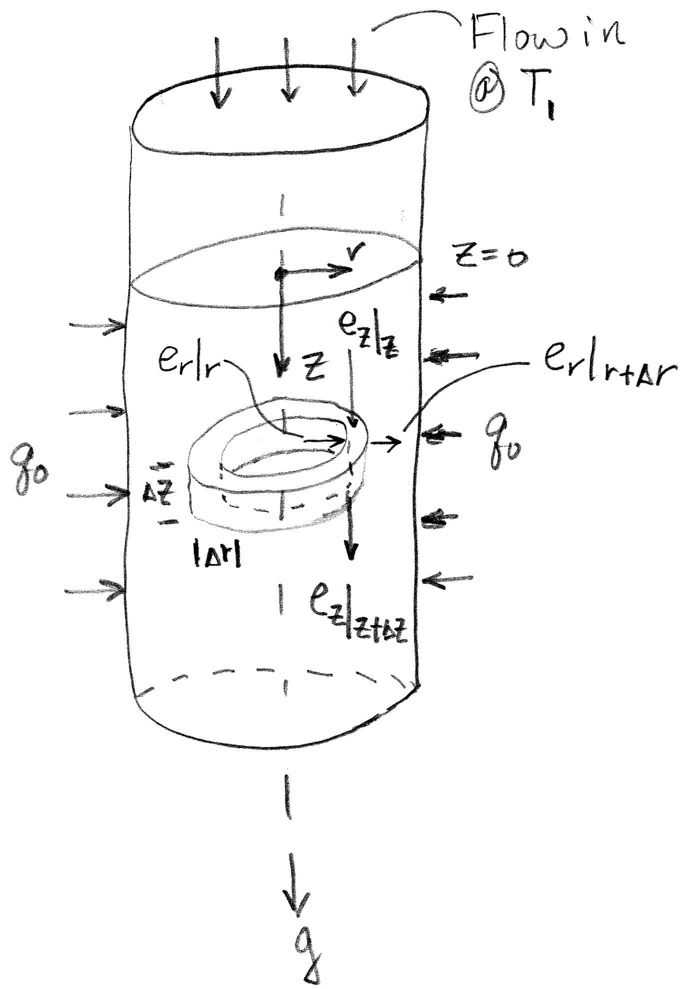
Forced Convection, 10.8



Assumptions:

- steady-state flow,
- constant ρ, μ, k
- $v_z = f(r)$ only
- $v_r = v_\theta = 0$
- $T = f(r,z)$ only.

Shell Energy Balance:

rate of energy in at r	$(2\pi r e_r)\|_r \,\Delta z$
" " " out " $r+\Delta r$	$(2\pi r e_r)\|_{r+\Delta r} \,\Delta z$
" " " in at z	$(2\pi r \Delta r)\, e_z\|_z$
" " " out " $z+\Delta z$	$(2\pi r \Delta r)\, e_z\|_{z+\Delta z}$
Work Done on Fluid By Gravity	$(2\pi r \Delta r L)\, v_z g_z \rho$

Summing, equating to 0, ÷ by $2\pi \Delta r \Delta z$

$$\frac{(re_r)|_r - (re_r)|_{r+\Delta r}}{\Delta r} + r\,\frac{e_z|_z - e_z|_{z+\Delta z}}{\Delta z} + \rho g_z v_z r = 0$$

lim as $\Delta r, \Delta z \to 0$.

$$-\frac{1}{r}\frac{\partial}{\partial r}(re_r) - \frac{\partial e_z}{\partial z} + \rho v_z g = 0$$

$$e_r = \delta_r \cdot e = \left(\tfrac{1}{2}\rho v^2 + \rho\hat{H}\right)\cancel{v_r}^{\,0} + [\tau\cdot v]_r + q_r$$

$$[\tau\cdot v]_r = \tau_{rr}\cancel{v_r}^{\,0} + \tau_{r\theta}\cancel{v_\theta}^{\,0} + \tau_{rz} v_z = -\mu\frac{\partial v_z}{\partial r} v_z$$

$$q_r = -k\frac{\partial T}{\partial r}$$

$$= -\left(\mu\frac{\partial v_z}{\partial r}\right)v_z - k\frac{\partial T}{\partial r}$$

$$e_z = \delta_z \cdot e = \left(\tfrac{1}{2}\rho v_z^2 + \rho\hat{H}\right)v_z + [\tau\cdot v]_z + q_z$$

$$\hat{H} = \hat{H}^\circ + \hat{C}_p(T - T^\circ) + \frac{1}{\rho}(p - p^\circ)$$ -- see pg. 286.

$$[\tau\cdot v]_z = \tau_{zr}\cancel{v_r}^{\,0} + \tau_{z\theta}\cancel{v_\theta}^{\,0} + \tau_{zz} v_z = -2\mu\left(\frac{\partial v_z}{\partial z}\right)v_z$$

$$q_z = -k\frac{\partial T}{\partial z}$$

$$= \left(\tfrac{1}{2}\rho v_z^2\right)v_z + \rho\hat{C}_p(T - T_0)v_z + (p - p^\circ)v_z - 2\mu\left(\frac{\partial v_z}{\partial z}\right)v_z - k\frac{\partial T}{\partial z}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(re_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-\left(\mu\frac{\partial v_z}{\partial r}\right)v_z - k\frac{\partial T}{\partial r}\right)\right)$$

$$\left\{ \begin{aligned} &= -\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-\left(\mu\frac{\partial v_z}{\partial r}\right)\right)\right)v_z - \frac{r}{r}\left(-\mu\frac{\partial v_z}{\partial r}\right)\frac{\partial v_z}{\partial r} \\ &\qquad\qquad\qquad -\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-k\frac{\partial T}{\partial r}\right)\right) \\ &= \mu \overset{v_z}{\downarrow}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \mu\left(\frac{\partial v_z}{\partial r}\right)^2 + k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \end{aligned} \right.$$

$$-\frac{\partial e_z}{\partial z} = -\frac{\partial}{\partial z}\left[\left(\frac{1}{2}\rho v_z^2\right)v_z + \rho\hat{C}_p(T-T^0)v_z + (p-p^0)v_z - 2\mu\left(\frac{\partial v_z}{\partial z}\right)v_z - k\frac{\partial T}{\partial z}\right]$$

$$= -\rho\hat{C}_p v_z\frac{\partial T}{\partial z} - v_z\frac{\partial P}{\partial z} + k\frac{\partial^2 T}{\partial z^2}$$

Combining Terms;

$$\rho\hat{C}_p v_z\frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \overbrace{\frac{\partial^2 T}{\partial z^2}}^{\text{small}}\right] + \overbrace{\mu\left(\frac{\partial v_z}{\partial r}\right)^2}^{\text{viscous heating } \sim 0} +$$

$$v_z\underbrace{\left[-\frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g\right]}_{\text{eqn. of motion (z-component)} = 0}$$

$$\boxed{\rho\hat{C}_p v_{z,max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]}$$

BC1	$r=0$	T is finite
BC2	$r=R$	$k\frac{\partial T}{\partial r} = q_0$ (constant)
BC3	$z=0$	$T = T_1$

Dimensionless Quantities.

$$\Theta = \frac{k(T-T_1)/R}{q_0} \quad , \quad \xi = \frac{r}{R} \quad , \quad \zeta = \frac{z}{\rho \hat{C}_p v_{z,max} R^2/k}$$

$$(1-\xi^2)\frac{\partial \Theta}{\partial \zeta} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \Theta}{\partial \xi}\right)$$

BC1 $\quad \xi = 0 \quad \Theta$ finite.

BC2 $\quad \xi = 1 \quad \dfrac{\partial \Theta}{\partial \xi} = 1$

BC3 $\quad \zeta = 0 \quad \Theta = 0$

Solution for large z, large ζ

$$\Theta(\xi, \zeta) = C_0 \zeta + \Psi(\xi) \qquad \text{is suggested}$$

Subst. into eqn. above;

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \Psi}{\partial \xi}\right) = C_0(1-\xi^2)$$

Integrate Twice;

$$\psi(\xi) = C_0\left(\frac{\xi^2}{4} - \frac{\xi^4}{16}\right) + C_1 \ln \xi + C_2$$

$$\therefore \Theta(\zeta, \xi) = C_0 \zeta + C_0\left(\frac{\xi^2}{4} - \frac{\xi^4}{16}\right) + C_1 \ln \xi + C_2 .$$

BC1 $\longrightarrow \quad C_1 = 0$

BC2 $\longrightarrow \quad C_0 = 4$

Condition 4: $2\pi R z q_0 = \int_0^{2\pi}\int_0^R \rho \hat{C}_p (T-T_1) v_z r\,dr\,d\theta$

or $\zeta = \int_0^1 \Theta(\zeta,\xi)(1-\xi^2)\xi\,d\xi$

$$= \int_0^1 \left(4\zeta + \xi^2 - \tfrac{1}{4}\xi^4 + C_2\right)(\xi - \xi^3)\,d\xi$$

$$= \int_0^1 \left(4\zeta\xi + \xi^3 - \tfrac{1}{4}\xi^5 + C_2\xi - 4\zeta\xi^3 - \xi^5 - \tfrac{1}{4}\xi^7 - C_2\xi^3\right)d\xi$$

$$= 2\zeta + \frac{1}{4} - \frac{1}{24} + \frac{C_2}{2} - \zeta - \frac{1}{6} + \frac{1}{32} - \frac{C_2}{4}$$

$$0 = \frac{1}{4} + \frac{1}{32} - \left(\frac{1}{24} + \frac{1}{6}\right) + \frac{C_2}{4}$$

$$0 = \frac{9}{32} - \frac{5}{24} + \frac{C_2}{4}$$

$$0 = \frac{27}{96} - \frac{20}{96} + \frac{C_2}{4}$$

$$C_2 = -\frac{7}{24}$$

$$\boxed{\Theta(\zeta,\xi) = 4\zeta + \xi^2 - \tfrac{1}{4}\xi^4 - 7/24}$$

exact for $\zeta \to \infty$

accurate to within 2% for $\zeta > 0.1$