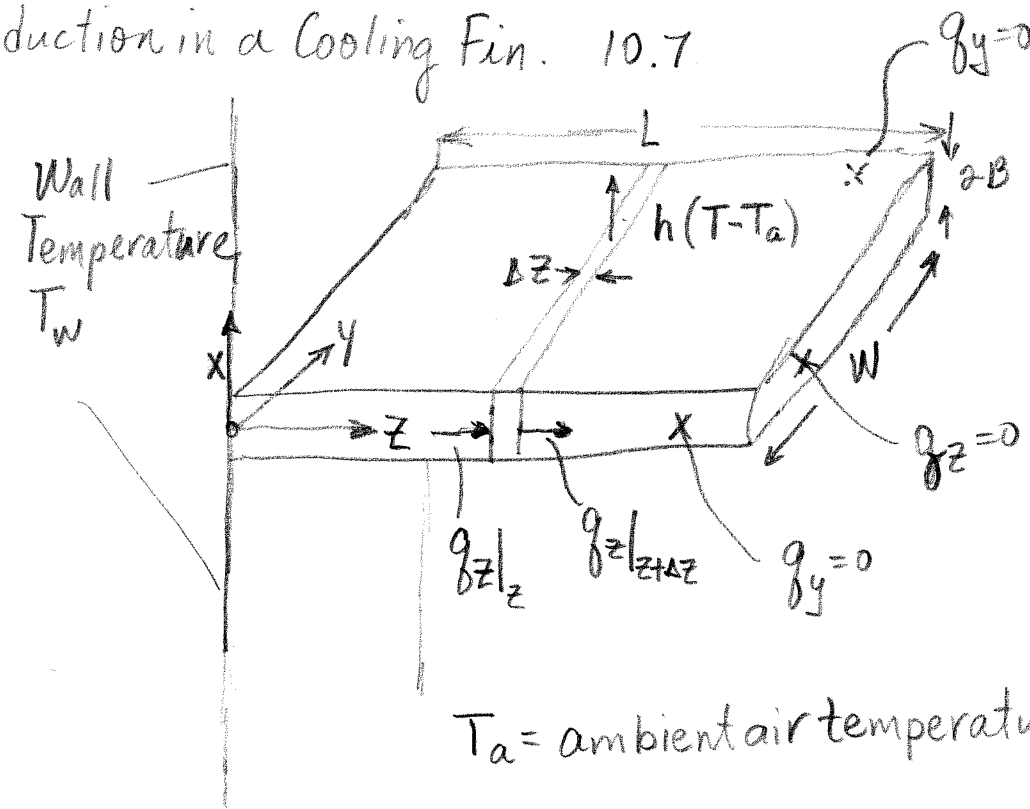
Heat Conduction in a Cooling Fin. 10.7



T_a = ambient air temperature

Assumptions
- $T = f(z)$ only (fin is very thin, so $T \neq f(x)$)
- k = constant
- heat is lost from the shell in the fin, $h(T - T_a)$ where h is a constant.

Energy Balance:

For conduction only, $v = 0$

$$e_z = q_z = -k\frac{dT}{dz}$$

rate of energy in at z	$(2BW)q_z\|_z$
" " " out " $z+\Delta z$	$(2BW)\,q_z\|_{z+\Delta z}$
rate of heat loss from shell	$(2W\Delta z)\,h(T-T_a)$

$$2BW q_z|_z - 2BW q_z|_{z+\Delta z} - 2W\Delta z\, h(T-T_a) = 0$$

$\div$ by $2BW\Delta z$, $\lim \Delta z \to 0$

$$-\frac{dq_z}{dz} = \frac{h}{B}(T-T_a)$$

Fourier's Law, $q_z = -k\frac{dT}{dz}$

$$\frac{d^2T}{dz^2} = \frac{h}{Bk}(T-T_a).$$

BC1 $z=0$ $T=T_w$

BC2 $z=L$ $dT/dz=0$

Let $\Theta = \frac{T-T_a}{T_w-T_a}$, $\zeta = \frac{z}{L}$, $N^2 = \frac{hL^2}{kB} = \frac{hL}{k}\frac{L}{B}$

$= Bi\left(\frac{L}{B}\right)$

Bi = Biot Number

$$\frac{d^2\Theta}{d\zeta^2} = N^2\Theta$$

BC1 $\zeta=0$ $\Theta=1$

BC2 $\zeta=1$ $d\Theta/d\zeta=0$

From Appendix C, pg 852

$$\mathrm{II} = C_1 \exp(N\zeta) + C_2 \exp(-N\zeta)$$

BC1 $\quad 1 = C_1(1) + C_2(1) \Rightarrow C_2 = 1 - C_1$

$$\frac{d\,\mathrm{II}}{d\zeta} = \frac{C_1}{N}\exp(N\zeta) - \frac{C_2}{N}\exp(-N\zeta)$$

BC2 $\quad 0 = \frac{C_1}{N}\exp(N) - \frac{C_2}{N}\exp(-N)$

$$0 = C_1 \exp(N) - (1-C_1)\exp(-N)$$

$$0 = C_1(\exp(N) + \exp(-N)) - \exp(-N)$$

$$C_1 = \frac{\exp(-N)}{\exp(N)+\exp(-N)}$$

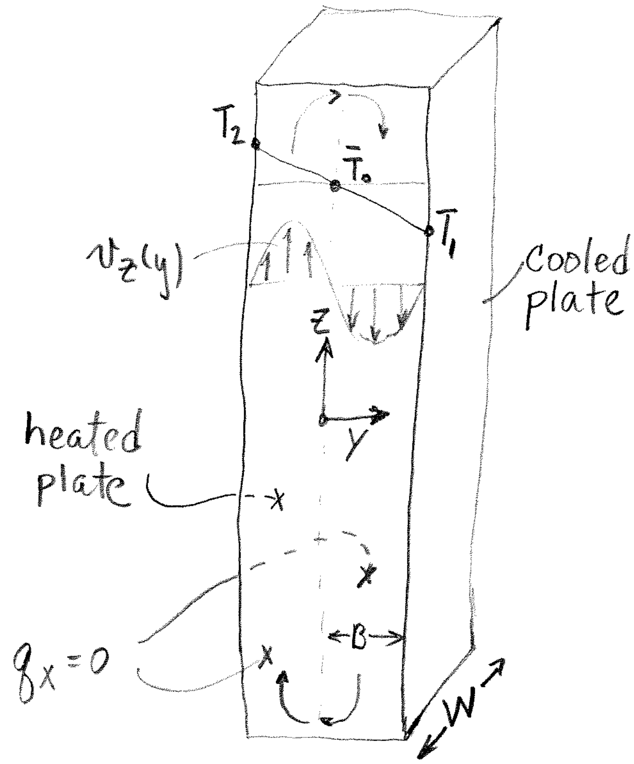
$$C_2 = 1 - C_1 = 1 - \frac{\exp(-N)}{\exp(N)+\exp(-N)}$$

$$= \frac{\exp(N)}{\exp(N)+\exp(-N)}$$

$$\mathrm{II} = \frac{\exp(-N)\exp(N\zeta)}{\exp(N)+\exp(-N)} + \frac{\exp(N)\exp(-N\zeta)}{\exp(N)+\exp(-N)}$$

$$= \frac{\exp(-N(1-\zeta)) + \exp(N(1-\zeta))}{\exp(N)+\exp(-N)} \boxed{= \frac{\cosh(N(1-\zeta))}{\cosh N}}$$

Free Convection Heat Transfer; 10.9



Assumptions:

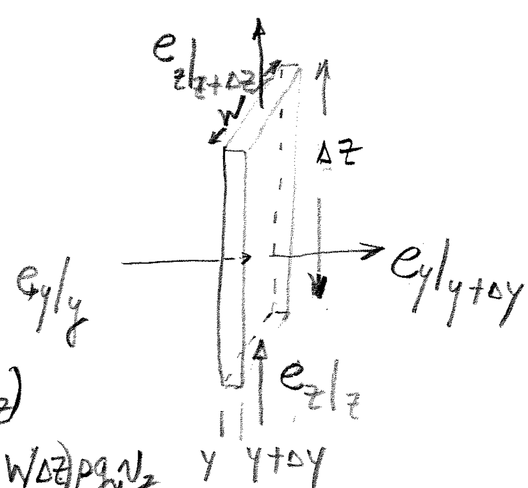
- steady-state for $v_z(y)$ and $T(y)$.
- $v_z = f(y)$ only.
- $T = f(y)$ only.
- μ, k constant.
- $\rho = \bar{\rho} - \bar{\rho}\,\bar{\beta}\,(T - \bar{T})$

where $\bar{T} = \frac{1}{2}(T_2 + T_1)$

$\bar{\beta} = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$ coefficient of volume expansion

$$= \frac{1}{1/\rho}\left(\frac{\partial(1/\rho)}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Shell Balance on Energy:



rate of energ in at y $(\Delta z W e_y|_y)$

" " " out " $y+\Delta y$ $(\Delta z W e_y|_{y+\Delta y})$

" " " in at $z=0$ $(\Delta y W e_z|_z)$

" " " out at $z=L$ $(\Delta y W e_z|_{z+\Delta z})$

rate of work on shell by gravity $(\Delta y W \Delta z)\rho g v_z$

$$(\Delta z W e_y|_y) - (\Delta z W e_y|_{y+\Delta y}) + (\Delta y W e_z|_z) - (\Delta y W e_z|_{z+\Delta z}) + (\Delta y W \Delta z)\rho g_z v_z = 0$$

$\div \Delta z W \Delta y$ let $\Delta z, \Delta y \to 0$

$$-\frac{\partial e_y}{\partial y} - \frac{\partial e_z}{\partial z} + \rho g_z v_z = 0$$

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$$e_y = \delta_y \cdot e = (\tfrac{1}{2}\rho v^2 + \rho \hat{H})\cancelto{0}{v_y} + [\tau \cdot v]_y + q_y$$

$$[\tau\cdot v]_y = \tau_{yx}\cancelto{0}{v_x} + \tau_{yy}\cancelto{0}{v_y} + \tau_{yz} v_z = -\mu\left(\frac{\partial v_z}{\partial y}\right) v_z$$

$$q_y = -k\frac{\partial T}{\partial y}$$

$$= -\mu\left(\frac{\partial v_z}{\partial y}\right) v_z - k\frac{\partial T}{\partial y}$$

$$-\frac{\partial e_y}{\partial y} = \mu\left(\frac{\partial v_z}{\partial y}\right)\frac{\partial v_z}{\partial y} + \mu v_z \frac{\partial^2 v_z}{\partial y^2} + k\frac{\partial^2 T}{\partial y^2}$$

$$e_z = \delta_z \cdot e = (\tfrac{1}{2}\rho v^2 + \rho\hat{H}) v_z + [\tau\cdot v]_z + q_z$$

$$v^2 = \cancelto{0}{v_x^2} + \cancelto{0}{v_y^2} + v_z^2 = v_z^2$$

$$\hat{H} = \hat{H}^\circ + \hat{C}_p(T - T^\circ) + \frac{1}{\rho}(p - p^\circ) \quad \text{(pg 286)}$$

$$[\tau\cdot v]_z = \tau_{zx}\cancelto{0}{v_x} + \tau_{zy}\cancelto{0}{v_y} + \tau_{zz} v_z = -2\mu\left(\frac{\partial v_z}{\partial z}\right) v_z + \overset{(\nabla\cdot v)}{0}$$

$$q_z = -k\cancelto{0}{\frac{\partial T}{\partial z}} \quad T = f(y) \text{ only}$$

$$= \frac{1}{2}\rho v_z^2 \cdot v_z + \rho\hat{H}^\circ v_z + \rho\hat{C}_p(T - T^\circ) v_z + (p - p^\circ) v_z - 2\mu\left(\frac{\partial v_z}{\partial z}\right) v_z$$

$$-\frac{\partial e_z}{\partial z} = 0 - 0 - 0 - 0 - v_z\frac{\partial p}{\partial z} - 0 - 0$$