Subst. for $e = (\frac{1}{2}\rho v^2 + \rho \hat{U})\mathbf{v} + [\pi \cdot \mathbf{v}] + q$

but $\pi = p\delta + \tau \longrightarrow [\pi \cdot \mathbf{v}] = p\mathbf{v} + [\tau \cdot \mathbf{v}]$

and $\nabla \cdot e = \nabla \cdot (\frac{1}{2}\rho v^2 + \rho \hat{U})\mathbf{v}) + \nabla \cdot p\mathbf{v} + \nabla \cdot [\tau \cdot \mathbf{v}]$

Wed. 3/15/06

Eqn. of Energy — Kinetic + Internal

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho \hat{U}) = &-(\nabla \cdot (\tfrac{1}{2}\rho v^2 + \rho \hat{U})\mathbf{v}) - (\nabla \cdot q) \\ &-(\nabla \cdot p\mathbf{v}) - (\nabla \cdot [\tau \cdot \mathbf{v}]) + \rho(\mathbf{v} \cdot g)\end{aligned}} \qquad (11.1\text{-}7)$$

does not include nuclear, radiative, electromagnetic or chemical sources of energy.

Including Potential Energy:

$$g = -\nabla \hat{\Phi} \qquad \hat{\Phi} = gh$$

$\rho(\mathbf{v} \cdot g)$ term:

$$\begin{aligned}\rho(\mathbf{v} \cdot g) &= -(\rho\mathbf{v} \cdot \nabla \hat{\Phi}) && \text{due to} \downarrow \\ &= -(\nabla \cdot \rho \mathbf{v} \hat{\Phi}) + \hat{\Phi}(\nabla \cdot \rho \mathbf{v}) && (A.4\text{-}19) \\ &= -(\nabla \cdot \rho \mathbf{v} \hat{\Phi}) + \hat{\Phi}\frac{\partial \rho}{\partial t} && (3.1\text{-}4) \\ &= -(\nabla \cdot \rho \mathbf{v} \hat{\Phi}) + \frac{\partial}{\partial t}(\rho \hat{\Phi}) && \hat{\Phi} \neq f(t)\end{aligned}$$

$$\boxed{\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho \hat{U} + \rho \hat{\Phi}) = -(\nabla \cdot (\tfrac{1}{2}\rho v^2 + \rho \hat{U} + \rho \hat{\Phi})\mathbf{v}) - (\nabla \cdot q) - (\nabla \cdot p\mathbf{v}) - (\nabla \cdot [\tau \cdot \mathbf{v}])}$$

Eqn. of Energy: Kinetic + Internal + Potential

Subtract Mechanical Energy Eqn (3.3-1) From Eqn 11.1-7, $\Rightarrow$ Internal Energy Eqn.

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) = -\left(\nabla\cdot\tfrac{1}{2}\rho v^2 v\right) - (\nabla\cdot p v) - p(-\nabla\cdot v) - (\nabla\cdot[\tau\cdot v]) - (\tau:\nabla v) + \rho(v\cdot g)$$

(3.3-1)

Eqn. of Change: Internal Energy

$$\boxed{\frac{\partial}{\partial t}(\rho\hat{U}) = -(\nabla\cdot\rho\hat{U}v) - p(-\nabla\cdot v) - (\tau:\nabla v) - (\nabla\cdot q)} \quad (11.2-1)$$

11.2-1

$-p(\nabla\cdot v)$ – reversible exchange of mechanical and internal energy: (+) or (−) depending on sign of $\nabla\cdot v$

expansion: $\nabla\cdot v$ (+)

compression: $\nabla\cdot v$ (−)

$-(\tau:\nabla v)$ – irreversible exchange of mechanical to internal energy. $-(\tau:\nabla v)$ (+)

since $\rho\frac{D\hat{U}}{Dt} = \frac{\partial}{\partial t}(\rho\hat{U}) + \nabla\cdot\rho v\hat{U}$

$$\boxed{\rho\frac{D\hat{U}}{Dt} = -(\nabla\cdot q) - p(\nabla\cdot v) - (\tau:\nabla v)}$$

11.2-2

using substantial derivative

Eqn. of Change: Enthalpy

$$\hat{H} = \hat{U} + p\hat{V} = \hat{U} + P/\rho$$

$$\therefore \frac{D\hat{U}}{Dt} = \frac{D\hat{H}}{Dt} - \frac{D}{Dt}\left(\frac{P}{\rho}\right)$$

$$= \frac{D\hat{H}}{Dt} - \frac{1}{\rho}\frac{DP}{Dt} + P/\rho^2 \frac{D\rho}{Dt}$$ eqn. (A) Table 3.5-1

$$= \frac{D\hat{H}}{Dt} - \frac{1}{\rho}\frac{DP}{Dt} + \frac{P}{\rho^2}\left(-\rho(\overline{\nabla}\cdot v)\right)$$

or $$\rho\frac{D\hat{U}}{Dt} = \rho\frac{D\hat{H}}{Dt} - \frac{DP}{Dt} - P(\overline{\nabla}\cdot v)$$

11.2-2 Becomes

$$\rho\frac{D\hat{U}}{Dt} = -(\nabla\cdot q) - P(\nabla\cdot v) - (\tau : \nabla v)$$

$$\rho\frac{D\hat{H}}{Dt} - \frac{Dp}{Dt} - \cancel{P(\nabla\cdot v)} = -(\nabla\cdot q) - \cancel{P(\overline{\nabla}\cdot v)} - (\tau : \nabla v)$$

$$\boxed{\rho\frac{D\hat{H}}{Dt} = -(\nabla\cdot q) - (\tau : \nabla v) + \frac{DP}{Dt}} \quad 11.2-3$$

Eqn. of Change: For Temperature.

From Eqn. 9.8-7 $\quad d\hat{H} = \left(\frac{\partial \hat{H}}{\partial T}\right)_p dT + \left(\frac{\partial \hat{H}}{\partial p}\right)_T dp$

$$= \hat{C}_p dT + \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right] dp$$

$$\therefore \rho \frac{D\hat{H}}{Dt} = \rho \hat{C}_p \frac{DT}{Dt} + \rho\left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right] \frac{Dp}{Dt}$$

$$= \rho \hat{C}_p \frac{DT}{Dt} + \left[1 + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p\right] \frac{Dp}{Dt}$$

Eqn 11.2-3 becomes.

$$\rho \frac{D\hat{H}}{Dt} = -(\nabla \cdot q) - (\tau : \nabla v) + \frac{Dp}{Dt}$$

$$\rho \hat{C}_p \frac{DT}{Dt} + \left[\cancel{1} + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p\right] \frac{Dp}{Dt} = -(\nabla \cdot q) - (\tau : \nabla v) + \cancel{\frac{Dp}{Dt}}$$

$$\boxed{\rho \hat{C}_p \frac{DT}{Dt} = -(\nabla \cdot q) - (\tau : \nabla v) - \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p \frac{Dp}{Dt}} \quad 11.2-5$$

① Ideal Gas, $(\partial \ln \rho / \partial \ln T)_p = -1$; $(\tau : \nabla v) \sim 0$
k constant, Fourier's Law $q = -k \nabla T$

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + \frac{Dp}{Dt}$$ or

$$\rho \hat{C}_v \frac{DT}{Dt} = k \nabla^2 T - p(\nabla \cdot v)$$

② Fluid Flow / Constant Pressure, $Dp/Dt = 0$.

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T$$

③ Fluid Flow / Constant ρ, $(\partial \ln \rho / \partial \ln T)_p = 0$

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T$$

④ Stationary Solid, $v = 0$

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = k \nabla^2 T$$ -- Conduction Eqn. developed by Fourier.

Fri 3/17/06

Table B.9 in Appendix B contains all terms