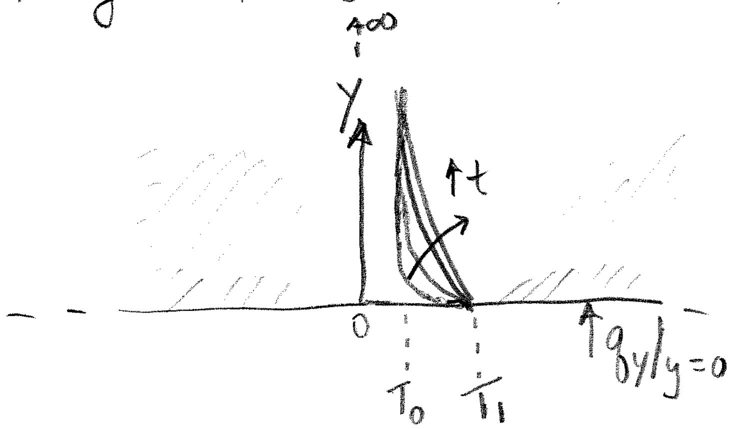
Chapter 12. Temperature Distributions in More Than One Independent Variable

12.1 Unsteady Heat Conduction in Solids



$T = f(t, y)$

$q_y|_{y=0}$ – heat flux @ $y=0$

From Table B.9

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + \cancelto{0}{v_x}\frac{\partial T}{\partial x} + \cancelto{0}{v_y}\frac{\partial T}{\partial y} + \cancelto{0}{v_z}\frac{\partial T}{\partial z}\right) = k\left[\cancelto{0}{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2} + \cancelto{0}{\frac{\partial^2 T}{\partial z^2}}\right] + \cancelto{0}{\mu \Phi_v}$$

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} \quad \text{or} \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}$$

IC	$t = 0$	all y	$T = T_0$
BC1	$t > 0$	$y = 0$	$T = T_1$
BC2	$t > 0$	$y \to \infty$	$T = T_0$

where

$$\alpha = \frac{k}{\rho \hat{C}_p} \quad (m^2/s)$$

"thermal diffusivity"

let $\Theta = \dfrac{T - T_0}{T_1 - T_0}$, $\Rightarrow$ $\dfrac{\partial \Theta}{\partial t} = \alpha \dfrac{\partial^2 \Theta}{\partial y^2}$ (12.1-3)

let $\eta = \dfrac{y}{\sqrt{4\alpha t}}$ $\Rightarrow$ $\dfrac{\partial \Theta}{\partial t} = \dfrac{\partial \Theta}{\partial \eta}\dfrac{\partial \eta}{\partial t} = -\dfrac{\eta}{2t}\dfrac{\partial \Theta}{\partial \eta}$

$$\frac{\partial \Theta}{\partial y} = \frac{\partial \Theta}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\alpha t}}\frac{\partial \Theta}{\partial \eta}$$

$$\frac{\partial^2 \Theta}{\partial y^2} = \frac{1}{4\alpha t}\frac{\partial^2 \Theta}{\partial \eta^2}$$

$\therefore$ (12.1-3) becomes,

$$\frac{d^2 \Theta}{d\eta^2} + 2\eta \frac{d\Theta}{d\eta} = 0$$

IC & BC2 $\Rightarrow$ $\eta = \infty$ $\Theta = 0$

BC1 $\Rightarrow$ $\eta = 0$ $\Theta = 1$

let $\psi = \dfrac{d\Theta}{d\eta}$ $\therefore$ (12.1-3) becomes $\dfrac{d\psi}{d\eta} + 2\eta\,\psi = 0$

integrating $\int \dfrac{d\psi}{\psi} = -2\int \eta\, d\eta \Rightarrow \ln\psi = -\eta^2 + \ln C_1$

$\Rightarrow \psi = \dfrac{d\Theta}{d\eta} = C_1 \exp(-\eta^2)$

integrating again; $\Theta = C_1 \int_0^{\eta} \exp(-\bar{\eta}^2)\, d\bar{\eta} + C_2$

applying BCs (as in Ch 4 + lecturenotes on pg 33.)

$$C_1 = -\frac{2}{\sqrt{\pi}} \quad , \quad C_2 = 1$$

$\therefore$ $\Theta = 1 - \text{erf}(\eta)$ or

$$\boxed{\frac{T - T_0}{T_1 - T_0} = 1 - \text{erf}\left(\frac{y}{\sqrt{4\alpha t}}\right)} \qquad (12.1\text{-}8)$$

Thermal Penetration Thickness, δ_T

min y where T has risen by $< 1\%$, $\frac{T - T_0}{T_1 - T_0} \leq 0.01$,

so $\text{erf}\left(\frac{y}{\sqrt{4\alpha t}}\right) \simeq .99 \longrightarrow$ occurs for $y/\sqrt{4\alpha t} \approx 2$

or $\frac{\delta_T}{\sqrt{4\alpha t}} = 2 \longrightarrow \boxed{\delta_T = 4\sqrt{\alpha t}}$

Wall Heat Flux, $q_y|_{y=0}$

$$q_y|_{y=0} = -k \left.\frac{dT}{dy}\right|_{y=0}$$

$$\left.\frac{dT}{dy}\right|_{y=0} = (T_1 - T_0) \left.\frac{d\Theta}{dy}\right|_{y=0} = (T_1 - T_0) \underbrace{\left.\frac{d\Theta}{d\eta}\right|}_{\eta=0} \frac{d\eta}{dy}$$

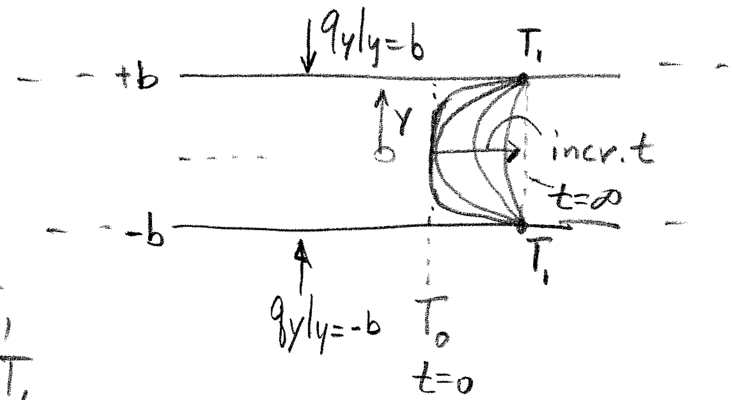
$$= (T_1 - T_0) \left.\frac{d\Theta}{d\eta}\right|_{\eta=0} \frac{1}{\sqrt{4\alpha t}}$$

$$\left.\frac{d\Theta}{d\eta}\right|_{\eta=0} = c_1 \exp(-\eta^2) = -\frac{2}{\sqrt{\pi}} \exp(-0^2) = -\frac{2}{\sqrt{\pi}}$$

$$\therefore\ q_y|_{y=0} = -k \left.\frac{dT}{dy}\right|_{y=0} = \frac{2k}{\sqrt{\pi}} \frac{(T_1 - T_0)}{\sqrt{4\alpha t}} \boxed{= \frac{k}{\sqrt{\pi \alpha t}} (T_1 - T_0)}$$

$t^{-1/2}$ dependence

Example 12.1-2



IC $t=0$, all y, $T=T_0$

BC1 $t>0$, $y=+b$, $T=T_1$

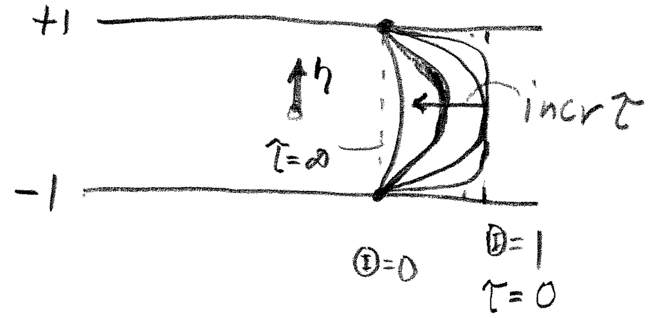
BC2 $t>0$, $y=-b$, $T=T_1$

dimensionless variables.

$$\Theta = \frac{T_1 - T}{T_1 - T_0},$$

$$\eta = \frac{y}{b}$$

$$\tau = \frac{\alpha t}{b^2}$$



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad \text{or}$$

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \eta^2}$$

IC $\tau = 0$, $\Theta = 1$

BC1,2 $\eta = \pm 1$ $\Theta = 0$, $\tau > 0$

$$\Theta(\tau, \eta) = \Theta_\infty(\eta) + \Theta_t(\tau, \eta)$$ "steady-state + transient".

steady-state:

$$\frac{\partial^2 \Theta}{\partial \eta^2} = 0 \qquad \eta = \pm 1, \quad \Theta = 0, \quad \tau = \infty$$

$$\Theta_\infty = C_1 \eta + C_2$$

BC1 $\quad 0 = C_1(1) + C_2$

BC2 $\quad 0 = -C_1(1) + C_2$

$$\left.\begin{array}{l} 0 = C_1(1) + C_2 \\ 0 = -C_1(1) + C_2 \end{array}\right\} C_1 = C_2 = 0$$

$$\boxed{\Theta_\infty = 0 \quad \text{for all } \eta}$$

Transient Solution:

(12.1-14) $\quad \dfrac{\partial \Theta_t}{\partial \tau} = \dfrac{\partial^2 \Theta}{\partial \eta^2}$ $\qquad$ IC ; $\Theta_t = 1, \quad \tau = 0$

BC1,2 ; $\Theta_t = 0, \quad \eta = \pm 1$

separation of variables:

$$\Theta_t(\tau, \eta) = g(\tau) f(\eta) \Rightarrow$$

$$\frac{\partial \Theta_t}{\partial \tau} = \frac{\partial}{\partial \tau}(g(\tau) f(\eta)) = f \frac{\partial g}{\partial \tau}$$

$$\frac{\partial^2 \Theta_t}{\partial \eta^2} = \frac{\partial}{\partial \eta^2}(g(\tau) f(\eta)) = g \frac{\partial^2 f}{\partial \eta^2}$$

(12.1-14) becomes $\quad f \dfrac{\partial g}{\partial \tau} = g \dfrac{\partial^2 f}{\partial \eta^2} \Rightarrow \dfrac{1}{g}\dfrac{\partial g}{\partial \tau} = \dfrac{1}{f}\dfrac{\partial^2 f}{\partial \eta^2} = -c^2$

$$\frac{dg}{d\tau} = -c^2 g \Rightarrow g = A \exp(-c^2 \tau)$$

$$\frac{d^2 f}{d\eta^2} + c^2 f = 0$$

$$f = B \sin c\eta + C \cos c\eta$$

Because of symmetry, $\Theta(\tau, \eta) = \Theta(\tau, -\eta)$

$$f(\eta) = f(-\eta) \quad , \quad B = 0$$

BC1 $\quad \Theta_t = gf = 0 \Rightarrow f(\eta = 1) = 0$

so $0 = C \cos c(1)$ or $\boxed{0 = C \cos c}$

therefore $c_n = (n + \frac{1}{2})\pi \quad , n = 0, \pm 1, \pm 2, \ldots \pm \infty$

$$\Theta_{t,n} = A_n C_n \exp[-(n+\tfrac{1}{2})^2 \pi^2 \tau] \cos(n+\tfrac{1}{2})\pi\eta$$

Adding all terms and noting $\Theta = \Theta_\infty + \Theta_t = 0 + \Theta_t$

or $\Theta_t = \Theta$

$$\Theta(\tau, \eta) = \sum_{n=0}^{\infty} D_n \exp[-(n+\tfrac{1}{2})^2 \pi^2 \tau] \cos(n+\tfrac{1}{2})\pi\eta$$

where $D_n = A_n C_n + A_{-(n+1)} C_{-(n+1)}$

IC. $\quad 1 = \sum_{n=0}^{\infty} D_n \cos(n+\frac{1}{2})\pi\eta$

$\times$ by $\cos(m+\frac{1}{2})\pi\eta$ and $\int_{-1}^{+1}$

$$\int_{-1}^{+1} \cos(m+\tfrac{1}{2})\pi\eta \, d\eta = \sum_{n=0}^{\infty} D_n \int_{-1}^{+1} \cos(m+\tfrac{1}{2})\pi\eta \cos(n+\tfrac{1}{2})\pi\eta \, d\eta$$

$$\left.\frac{\sin(m+\frac{1}{2})\pi\eta}{(m+\frac{1}{2})\pi}\right|_{\eta=-1}^{\eta=1} = D_m \left.\frac{\frac{1}{2}(m+\frac{1}{2})\pi\eta + \frac{1}{4}\sin 2(m+\frac{1}{2})\pi\eta}{(m+\frac{1}{2})\pi}\right|_{\eta=-1}^{\eta=+1}$$

$$D_m = \frac{2(-1)^m}{(m+\frac{1}{2})\pi}$$

$$\Theta(\tau,\eta) = \sum_{n=0}^{\infty} \frac{2(-1)^n}{(n+\frac{1}{2})\pi} \exp\left[-(n+\tfrac{1}{2})^2\pi^2\tau\right] \cos(n+\tfrac{1}{2})\pi\eta$$

$$\boxed{\frac{T_1-T}{T_1-T_0} = \sum_{n=0}^{\infty} \frac{2(-1)^n}{(n+\frac{1}{2})\pi} \exp\left[-(n+\tfrac{1}{2})^2\pi^2\frac{\alpha t}{b^2}\right] \cos(n+1)\pi\frac{y}{b}}$$

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