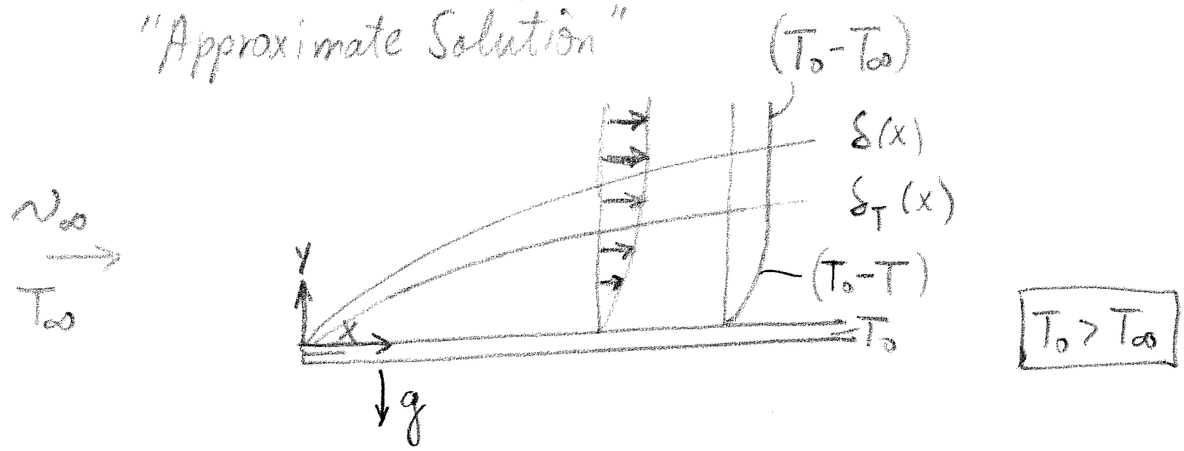
Boundary Layer Theory For Non-Isothermal Flows:

"Approximate Solution"



Equations of Change:

Continuity $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Motion: $$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = v_e \cancel{\frac{\partial v_e}{\partial x}}^{\,0} + \mu \frac{\partial^2 v_x}{\partial y^2} + \rho \cancel{g_x}^{\,0} \beta (T - T_\infty)$$

Energy: $$\rho \hat{C}_p\left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right) = k \frac{\partial^2 T}{\partial y^2} + \mu \cancel{\left(\frac{\partial v_x}{\partial y}\right)^2}^{\,0}$$

Intuition about velocity Field:

$v_x = 0$ $\quad$ $y = 0$ $\quad$ → no slip condition

$\frac{\partial v_x}{\partial y} = 0$ $\quad$ $y = \delta(x)$ $\quad$ → intuition

$\frac{\partial^2 v_x}{\partial y^2} = 0$ $\quad$ $y = 0$ $\quad$ → from eqn. of motion when $v_x = v_y = 0$, $y = 0$

an equation (polynomial) that will fit these BCs is:

$$v_x = v_\infty\left(2\left(\frac{y}{\delta(x)}\right) - 2\left(\frac{y}{\delta(x)}\right)^3 + \left(\frac{y}{\delta(x)}\right)^4\right)$$

<u>proof</u>: at $y=0$, $v_x = 0$

$$y=\delta(x), \quad \frac{\partial v_x}{\partial y} = v_\infty\left(2\left(\frac{1}{\delta(x)}\right) - 6\left(\frac{y^2}{\delta(x)^3}\right) + 4\left(\frac{y^3}{\delta(x)^4}\right)\right)\Bigg|_{y=\delta(x)}$$

$$= v_\infty\left(\frac{2}{\delta(x)} - \frac{6}{\delta(x)} + \frac{4}{\delta(x)}\right) = 0$$

$$y=0, \quad \frac{\partial^2 v_x}{\partial y^2} = v_\infty\left(-\frac{12y}{\delta^3_{(x)}} + \frac{12y^2}{\delta^4_{(x)}}\right) = 0$$

From Continuity Eqn:

$$\frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x} \Rightarrow \int_0^{v_y} \partial v_y = -\int_0^y \left(\frac{\partial v_x}{\partial x}\right) dy$$

$$\Rightarrow v_y = -\int_0^y \left(\frac{\partial v_x}{\partial x}\right) dy.$$

but $v_x = v_\infty\left(2\left(\frac{y}{\delta(x)}\right) - 2\left(\frac{y}{\delta(x)}\right)^3 + \left(\frac{y}{\delta(x)}\right)^4\right)$

$$\frac{\partial v_x}{\partial x} = v_\infty \frac{\partial}{\partial x}\left(2\left(\frac{y}{\delta(x)}\right) - 2\left(\frac{y}{\delta(x)}\right)^3 + \left(\frac{y}{\delta(x)}\right)^4\right)$$

$$= v_\infty \frac{\partial \delta(x)}{\partial x} \frac{\partial}{\partial \delta(x)}\left(2\left(\frac{y}{\delta(x)}\right) - 2\left(\frac{y}{\delta(x)}\right)^3 + \left(\frac{y}{\delta(x)}\right)^4\right)$$

$$\boxed{= v_\infty \frac{\partial \delta(x)}{\partial x}\left(-2\frac{y}{\delta^2_{(x)}} + 6\left(\frac{y^3}{\delta^4_{(x)}}\right) - 4\left(\frac{y^4}{\delta^5_{(x)}}\right)\right)}$$

$$v_y = -\int_0^y \left(\frac{\partial v_x}{\partial x}\right) dy$$

$$= -\int_0^y v_\infty \left(\frac{\partial \delta(x)}{\partial x}\right)\left(-2\frac{y}{\delta_{(x)}^2} + 6\left(\frac{y^3}{\delta_{(x)}^4}\right) - 4\left(\frac{y^4}{\delta_{(x)}^5}\right)\right) dy$$

$$\boxed{= -v_\infty \frac{\partial \delta(x)}{\partial x}\left(-\left(\frac{y}{\delta(x)}\right)^2 + \frac{6}{4}\left(\frac{y}{\delta(x)}\right)^4 - \frac{4}{5}\left(\frac{y}{\delta(x)}\right)^5\right)}$$

Eqn. of Motion can be $\int_0^{\delta(x)}$

$$\int_0^{\delta(x)} \rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) dy = \mu \int_0^{\delta} \frac{\partial}{\cancel{\partial y}}\left(\frac{\partial v_x}{\partial y}\right) \cancel{dy}$$

$$= -\mu \left.\frac{\partial v_x}{\partial y}\right|_{y=0}$$

$$\int_0^{\delta(x)} \rho \Bigg(v_\infty\left(2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\right)\left(v_\infty \frac{\partial \delta}{\partial x}\left(-2\frac{y}{\delta^2} + 6\frac{y^3}{\delta^4} - 4\frac{y^4}{\delta^5}\right)\right) +$$

$$\left(-v_\infty \frac{\partial \delta}{\partial x}\left(-\left(\frac{y}{\delta}\right)^2 + \frac{6}{4}\left(\frac{y}{\delta}\right)^4 - \frac{4}{5}\left(\frac{y}{\delta}\right)^5\right)\right)\left(v_\infty\left(\frac{2}{\delta} - 6\frac{y^2}{\delta^3} + 4\frac{y^3}{\delta^4}\right)\right)\Bigg) dy =$$

$$-\mu v_\infty \frac{2}{\delta}$$

$$\int_0^{\delta(x)} v_\infty^2 \frac{\partial \delta}{\partial x}\Bigg(-4\frac{y^2}{\delta^3} + 12\frac{y^4}{\delta^5} - 8\frac{y^5}{\delta^6} + 4\frac{y^4}{\delta^5} - 12\frac{y^6}{\delta^7} + 8\frac{y^7}{\delta^8} - 2\frac{y^5}{\delta^6} + 6\frac{y^7}{\delta^8} - 4\frac{y^8}{\delta^9}$$

$$+ 2\frac{y^2}{\delta^3} - 6\frac{y^4}{\delta^5} + 4\frac{y^5}{\delta^6} - 3\frac{y^4}{\delta^5} + 9\frac{y^6}{\delta^7} - 6\frac{y^7}{\delta^8} + \frac{8}{5}\frac{y^5}{\delta^6} - \frac{24}{5}\frac{y^7}{\delta^8} + \frac{16}{5}\frac{y^8}{\delta^9}\Bigg) dy =$$

$$-\frac{2\mu v_\infty}{\rho \delta}$$

$$v_\infty^2 \frac{\partial \delta}{\partial x}\left(-\frac{4}{3}+\frac{12}{5}-\frac{8}{6}+\frac{4}{5}-\frac{12}{7}+1-\frac{1}{3}+\frac{3}{4}-\frac{4}{9}\right.$$
$$\left.+\frac{2}{3}-\frac{6}{5}+\frac{2}{3}-\frac{3}{5}+\frac{9}{7}-\frac{3}{4}+\frac{8}{30}-\frac{3}{5}+\frac{16}{45}\right)=-\frac{2\mu v_\infty}{\rho\delta}$$

$$.11746\, v_\infty^2 \frac{\partial \delta}{\partial x} = \frac{2\mu v_\infty}{\rho \delta}$$

$$\delta\, d\delta = 17.03 \frac{\mu}{\rho v_\infty} dx$$

$$\int_0^\delta \delta\, d\delta = 17.03 \frac{\mu}{\rho v_\infty}\int_0^x dx$$

$$\delta^2 = 34.05 \frac{\mu x}{\rho v_\infty} \Rightarrow \boxed{\delta(x) = \sqrt{\frac{1260}{37}\frac{\nu x}{v_\infty}}}$$

Energy Eqn:

$$\rho \hat{C}_p\left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right) = k \frac{\partial^2 T}{\partial y^2}$$

Temperature distribution must satisfy.

$$\frac{T_0 - T}{T_0 - T_\infty} = 0 \quad \text{at} \quad y = 0 \quad \rightarrow \quad \text{boundary condition}$$

$$\frac{\partial T}{\partial y} = 0 \qquad y = \delta_T \longrightarrow \text{intuition}$$

$$\frac{\partial^2 T}{\partial y^2} = 0 \qquad y = 0 \longrightarrow$$ from energy eqn when $v_x = v_y = 0$ at $y = 0$.

an equation matching these conditions can be proven to be. (show yourselves).

$$\frac{T_0 - T}{T_0 - T_\infty} = 2\left(\frac{y}{\delta_T}\right) - 2\left(\frac{y}{\delta_T}\right)^3 + \left(\frac{y}{\delta_T}\right)^4 \qquad 0 \le y \le \delta_T$$

$$= 1 \qquad y \ge \delta_T$$

Eqn of Energy $\int_0^{\delta_T}$

$$\int_0^{\delta_T} \rho \hat{C}_p \left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} \right) dy = \int_0^{\delta_T} k \frac{\partial^2 T}{\partial y^2}\, dy.$$

$$\int_0^{\delta_T} \rho \hat{C}_p \Big(v_\infty \Big(2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\Big)\Big(\frac{\partial \delta_T}{\partial x}\Big(-2\frac{y}{\delta_T^2} + 6\frac{y^3}{\delta_T^4} - 4\frac{y^4}{\delta_T^5}\Big)\Big) +$$

$$\Big(-v_\infty \frac{1}{\Delta}\frac{\partial \delta_T}{\partial x}\Big(-\left(\frac{y}{\delta}\right)^2 + \frac{6}{4}\left(\frac{y}{\delta}\right)^4 - \frac{4}{5}\left(\frac{y}{\delta}\right)^5\Big)\Big(\frac{2}{\delta_T} - 6\frac{y^2}{\delta_T^3} + 4\frac{y^3}{\delta_T^4}\Big)\Big)\, dy$$

$$= -k\frac{2}{\delta_T}$$

$$\int_0^{\delta_T} \rho \hat{C}_p \Big[v_\infty \frac{\partial \delta_T}{\partial x} \Big(-4\frac{y^2}{\delta \delta_T^2} + 12\frac{y^4}{\delta \delta_T^4} - 8\frac{y^5}{\delta \delta_T^5} + 4\frac{y^4}{\delta^3 \delta_T^2} - 12\frac{y^6}{\delta^3 \delta_T^4} + 8\frac{y^7}{\delta^3 \delta_T^5}$$

$$-2\frac{y^5}{\delta^4 \delta_T^2} + 6\frac{y^7}{\delta^4 \delta_T^4} - 4\frac{y^8}{\delta^4 \delta_T^5} \Big) +$$

$$(-v_\infty) \frac{1}{\Delta} \frac{\partial \delta_T}{\partial x} \Big(-2\frac{y^2}{\delta^2 \delta_T} + 6\frac{y^4}{\delta^2 \delta_T^3} - 4\frac{y^5}{\delta^2 \delta_T^4} + 3\frac{y^4}{\delta^4 \delta_T}$$

$$-9\frac{y^6}{\delta^4 \delta_T^3} + 6\frac{y^7}{\delta^4 \delta_T^4} - \frac{8}{5}\frac{y^5}{\delta^5 \delta_T} + \frac{24}{5}\frac{y^7}{\delta^5 \delta_T^3} - \frac{16}{5}\frac{y^8}{\delta^5 \delta_T^4} \Big) \Big] dy$$

$$= -k\frac{2}{\delta_T}$$

$$\rho \hat{C}_p v_\infty \frac{\partial \delta_T}{\partial x} \Big(-\frac{4}{3}\Delta + \frac{12}{5}\Delta - \frac{4}{3}\Delta + \frac{4}{5}\Delta^3 - \frac{12}{7}\Delta^3 + \Delta^3 - \frac{1}{3}\Delta^4 + \frac{3}{4}\Delta^4 - \frac{4}{9}\Delta^4$$

$$-\frac{1}{\Delta}\Big(-\frac{2}{3}\Delta^2 + \frac{6}{5}\Delta^2 - \frac{2}{3}\Delta^2 + \frac{3}{5}\Delta^4 - \frac{9}{7}\Delta^4 + \frac{3}{4}\Delta^4 - \frac{4}{15}\Delta^5 + \frac{6}{10}\Delta^5 - \frac{16}{45}\Delta^5 \Big)$$

$$= -k\frac{2}{\delta_T}$$

$$\rho \hat{C}_p v_\infty \frac{\partial \delta_T}{\partial x} \Big(-\frac{4}{15}\Delta + \frac{3}{35}\Delta^3 - \frac{1}{36}\Delta^4 + \frac{2}{15}\Delta - \frac{9}{140}\Delta^3 + \frac{4}{180}\Delta^4 \Big) = -\frac{2k}{\delta_T}$$

$$\rho \hat{C}_p v_\infty \frac{\partial \delta_T}{\partial x} \Big(\frac{2}{15}\Delta - \frac{3}{140}\Delta^3 + \frac{1}{180}\Delta^4 \Big) = \frac{2k}{\delta_T}$$

$$\delta_T\, \partial \delta_T = \frac{2k}{\rho \hat{C}_p v_\infty\left(\frac{2}{15}\Delta - \frac{3}{140}\Delta^3 + \frac{1}{180}\Delta^4\right)}\,dx$$

$$\boxed{\delta_T = \sqrt{\frac{4}{\left(\frac{2}{15}\Delta - \frac{3}{140}\Delta^3 + \frac{1}{180}\Delta^4\right)}\left(\frac{\alpha x}{v_\infty}\right)}}$$

Δ^2

$$\left(\frac{\delta_T}{\delta}\right)^2 = \frac{\dfrac{4}{\left(\frac{2}{15}\Delta - \frac{3}{140}\Delta^3 + \frac{1}{180}\Delta^4\right)}\left(\dfrac{\alpha x}{v_\infty}\right)}{\dfrac{1260}{37}\left(\dfrac{\nu x}{v_\infty}\right)}$$

$$\frac{2}{15}\Delta^3 - \frac{3}{140}\Delta^5 + \frac{1}{180}\Delta^6 = \frac{37}{315} P_r$$

where $P_r = \dfrac{\nu}{\alpha}$

"Prandtl No."

solve for Δ, $\boxed{\Delta \sim P_r^{-1/3}}$ for $\Delta < 1$

for H_2O @ 300K, $P_r = 6.02$. $\Rightarrow$ $\Delta = 0.55$

Temperature Profile:

$$\frac{T_0 - T}{T_0 - T_\infty} = 2\left(\frac{y}{\delta_T}\right) - 2\left(\frac{y}{\delta_T}\right)^3 + \left(\frac{y}{\delta_T}\right)^4$$

$$= 2\left(\frac{y}{\Delta\delta}\right) - 2\left(\frac{y}{\Delta\delta}\right)^3 + \left(\frac{y}{\Delta\delta}\right)^4$$

$$\Delta = Pr^{-1/3}$$

$$\delta(x) = \sqrt{\frac{1260}{37}\frac{\nu x}{v_\infty}}$$