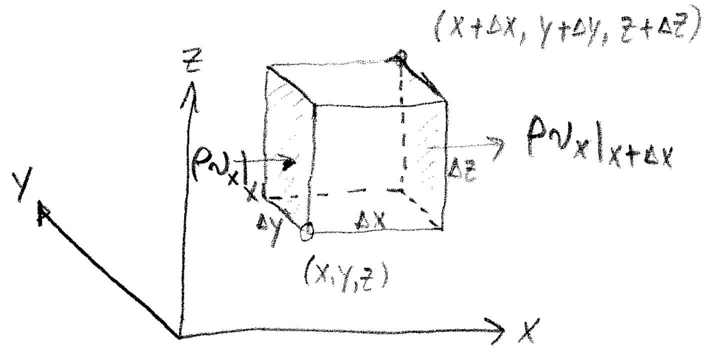
Ch 3. Eqn. of Change.

3.1 Eqn. of Continuity:



volume element
$(\Delta x\, \Delta y\, \Delta z) \equiv \Delta V$

rate of increase of mass = rate of mass in − rate of mass out.

$$(\Delta V)\frac{\partial \rho}{\partial t} = \Delta y \Delta z (\rho v_x)_x + \Delta x \Delta z (\rho v_y)|_y + \Delta x \Delta y (\rho v_z)|_z$$
$$- \Delta y \Delta z (\rho v_x)_{x+\Delta x} - \Delta x \Delta z (\rho v_y)_{y+\Delta y} - \Delta x \Delta y (\rho v_z)_{z+\Delta z}$$

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = -\Delta y \Delta z \left((\rho v_x)_{x+\Delta x} - (\rho v_x)_x\right) - \Delta x \Delta z \left((\rho v_y)_{y+\Delta y} - (\rho v_y)_y\right)$$
$$- \Delta x \Delta y \left((\rho v_z)_{z+\Delta z} - (\rho v_z)_z\right)$$

÷ by $\Delta x \Delta y \Delta z$ let $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$.

$$\boxed{\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right)} \quad 3.1\text{-}3$$

$$\boxed{\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho v)} \quad 3.1\text{-}4$$

where ∇ = vector differential oper.
$\equiv \delta_x \frac{\partial}{\partial x} + \delta_y \frac{\partial}{\partial y} + \delta_z \frac{\partial}{\partial z}$
δ_i = unit vectors

$\mathbf{V}$ = velocity vector $= \delta_x v_x + \delta_y v_y + \delta_z v_z$

$\rho = \rho(x,y,z,t)$ compressible fluids.

ρ = constant, incompressible

3.1-3 expand it.

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right)$$

$$= -\left(\rho \frac{\partial v_x}{\partial x} + v_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial v_y}{\partial y} + v_y \frac{\partial \rho}{\partial y} + \rho \frac{\partial v_z}{\partial z} + v_z \frac{\partial \rho}{\partial z}\right)$$

$$= -\left(\rho\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] + \left[v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}\right]\right)$$

$$= -\rho(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla \rho)$$

$$\boxed{\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho = -\rho(\nabla \cdot \mathbf{V})}$$

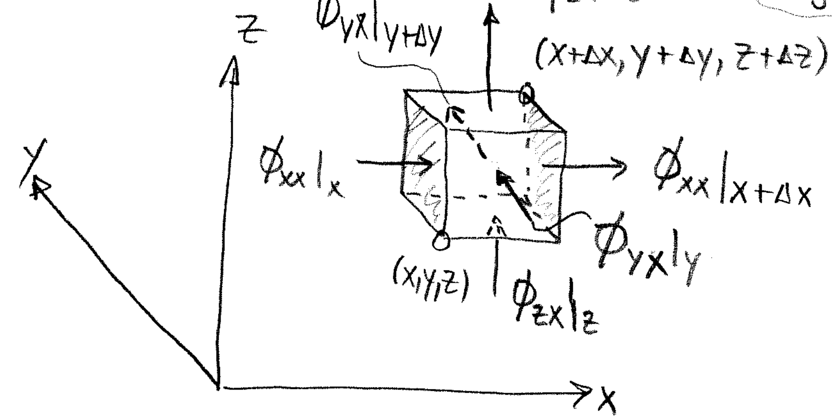
3.2 Eqn. of Motion (Conservation of Momentum)

$$\begin{array}{c} \text{rate of} \\ \text{increase of} \\ \text{momentum} \end{array} = \begin{array}{c} \text{rate of} \\ \text{momentum} \\ \text{in} \end{array} - \begin{array}{c} \text{rate of} \\ \text{momentum} \\ \text{out} \end{array} + \begin{array}{c} \text{external} \\ \text{force on} \\ \text{fluid.} \end{array}$$

Momentum is a vector quantity, having magnitude + direction. Momentum has components in 3 coordinate directions, (x, y, z in Cartesian coordinates).

- ϕ_{ij} are internal fluxes
- gravity is an example of an external force acting on the fluid elem.

x-momentum,



$$(\Delta x \Delta y \Delta z)\frac{\partial}{\partial t}(\rho v_x) = \Delta y \Delta z\, \phi_{xx}|_x + \Delta x \Delta z\, \phi_{yx}|_y + \Delta x \Delta y\, \phi_{zx}|_x$$
$$- \Delta y \Delta z\, \phi_{xx}|_{x+\Delta x} - \Delta x \Delta z\, \phi_{yx}|_{y+\Delta y} - \Delta x \Delta y\, \phi_{zx}|_{z+\Delta z}$$
$$+ \Delta x \Delta y \Delta z\, \rho g_x$$
$$= -\Big[\Delta y \Delta z(\phi_{xx}|_{x+\Delta x} - \phi_{xx}|_x) + \Delta x \Delta z(\phi_{yx}|_{y+\Delta y} - \phi_{yx}|_y)$$
$$+ \Delta x \Delta y(\phi_{zx}|_{z+\Delta z} - \phi_{zx}|_z)\Big] + \Delta x \Delta y \Delta z\, \rho g_x$$

$\div$ by $\Delta x \Delta y \Delta z$ $\quad \lim \Delta x \to 0,\ \Delta y \to 0,\ \Delta z \to 0$

$$\frac{\partial}{\partial t}(\rho v_x) = -\left(\frac{\partial \phi_{xx}}{\partial x} + \frac{\partial \phi_{yx}}{\partial y} + \frac{\partial \phi_{zx}}{\partial z}\right) + \rho g_x \qquad (3.2\text{-}4)$$

Similarly for y- and z-momentum components.

$$\frac{\partial}{\partial t}(\rho v_y) = -\left(\frac{\partial \phi_{xy}}{\partial x} + \frac{\partial \phi_{yy}}{\partial y} + \frac{\partial \phi_{zy}}{\partial z}\right) + \rho g_y \qquad (3.2\text{-}5)$$

$$\frac{\partial}{\partial t}(\rho v_z) = -\left(\frac{\partial \phi_{xz}}{\partial x} + \frac{\partial \phi_{yz}}{\partial y} + \frac{\partial \phi_{zz}}{\partial z}\right) + \rho g_z \qquad (3.2\text{-}6)$$

or $$\frac{\partial}{\partial t}(\rho v_i) = -[\nabla \cdot \phi]_i + \rho g_i \qquad (3.2\text{-}7)$$

where $i \equiv x, y,$ or z component

v_i = i- component of $\mathbf{v}$ vector

$[\nabla \cdot \phi]_i$ = i-component of $[\nabla \cdot \phi]$ vector

g_i = i- component of g vector.

When each component in 3.2-7 is multiplied by δ_i (unit vector in ith coordinate direction and all components added, we get the vector equivalent of 3.2-7.

$$\frac{\partial}{\partial t}(\rho v) = -[\nabla \cdot \phi] + \rho g$$

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But $\phi = \rho vv + \pi = \rho vv + p\delta + \tau$ (all tensors)

$$\boxed{\frac{\partial}{\partial t}(\rho v) = -[\nabla \cdot \rho vv] - \nabla p - [\nabla \cdot \tau] + \rho g} \qquad (3.2\text{-}9)$$