3.5 Eqns. of Change in Terms of Substantial Derivative.

Different Time Derivatives. $c = c(x, y, z, t)$ fish conc. in river.

① partial time derivative, $(\partial/\partial t)_{x,y,z}$

at one location, (x, y, z) in river, observe the rate of change of fish conc. with time.

$$\left(\frac{\partial c}{\partial t}\right)_{x,y,z}$$

② total time derivative, dc/dt

in a submarine motoring through the river with velocity components, dx/dt, $\frac{dy}{dt}$, $\frac{dz}{dt}$.

$$\frac{dc}{dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} + \frac{dx}{dt}\left(\frac{\partial c}{\partial x}\right)_{y,z,t} + \frac{dy}{dt}\left(\frac{\partial c}{\partial y}\right)_{x,z,t} + \frac{dz}{dt}\left(\frac{\partial c}{\partial z}\right)_{x,y,t}$$

③ substantial derivative, Dc/Dt

you are moving with the river with river velocity components v_x, v_y, v_z.

$$\frac{Dc}{Dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} + v_x\left(\frac{\partial c}{\partial x}\right)_{y,z,t} + v_y\left(\frac{\partial c}{\partial y}\right)_{x,z,t} + v_z\left(\frac{\partial c}{\partial z}\right)_{x,y,t}$$

$$\boxed{\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v \cdot \nabla c} \qquad (3.5\text{-}2)$$

Continuity Eqn in Terms of D/Dt:

eqn 3.1-3

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right)$$

$$= -\left(\rho\frac{\partial v_x}{\partial x} + v_x\frac{\partial \rho}{\partial x} + \rho\frac{\partial v_y}{\partial y} + v_y\frac{\partial \rho}{\partial y} + \rho\frac{\partial v_z}{\partial z} + v_z\frac{\partial \rho}{\partial z}\right)$$

$$= -\left(\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + \left(v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z}\right)\right)$$

$$= -\rho(\nabla\cdot v) - (v\cdot\nabla\rho)$$

or $$\frac{\partial \rho}{\partial t} + v\cdot\nabla\rho \equiv \boxed{\frac{D\rho}{Dt} = -\rho(\nabla\cdot v)}$$ Table 3.5-1 (A)

in compression, $\nabla\cdot v < 0$ so $D\rho/Dt > 0$, $\rho\uparrow$

for expansion, $\nabla\cdot v > 0$ so $D\rho/Dt < 0$, $\rho\downarrow$

Eqn. of Motion in Terms of D/Dt

$$\frac{\partial}{\partial t}(\rho v) = -[\nabla\cdot\rho v v] - \nabla p - [\nabla\cdot\tau] + \rho g$$

look at $\nabla\cdot\rho vv$ term.

$$\nabla\cdot\rho vv = \left[\frac{\partial}{\partial x}(\rho v_x v_x) + \frac{\partial}{\partial y}(\rho v_y v_x) + \frac{\partial}{\partial z}(\rho v_z v_x)\right]\delta_x$$
$$\left[\frac{\partial}{\partial x}(\rho v_x v_y) + \frac{\partial}{\partial y}(\rho v_y v_y) + \frac{\partial}{\partial z}(\rho v_z v_y)\right]\delta_y$$
$$\left[\frac{\partial}{\partial x}(\rho v_x v_z) + \frac{\partial}{\partial y}(\rho v_y v_z) + \frac{\partial}{\partial z}(\rho v_z v_z)\right]\delta_z$$

$$= \left(\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + \rho v_z \frac{\partial v_x}{\partial z}\right)\delta_x + \left(\rho v_x \frac{\partial v_y}{\partial x} + \rho v_y \frac{\partial v_y}{\partial y} + \rho v_z \frac{\partial v_y}{\partial z}\right)\delta_y$$
$$+ \left(\rho v_x \frac{\partial v_z}{\partial x} + \rho v_y \frac{\partial v_z}{\partial y} + \rho v_z \frac{\partial v_z}{\partial z}\right)\delta_z + \left(v_x \frac{\partial}{\partial x}(\rho v_x) + v_x \frac{\partial}{\partial y}(\rho v_y) + v_x \frac{\partial}{\partial z}(\rho v_z)\right)\delta_x$$
$$+ \left(v_y \frac{\partial}{\partial x}(\rho v_x) + v_y \frac{\partial}{\partial y}(\rho v_y) + v_y \frac{\partial}{\partial z}(\rho v_z)\right)\delta_y +$$
$$\left(v_z \frac{\partial}{\partial x}(\rho v_x) + v_z \frac{\partial}{\partial y}(\rho v_y) + v_z \frac{\partial}{\partial z}(\rho v_z)\right)\delta_z$$

$$= \rho v \cdot \nabla v + v(\nabla \cdot \rho v)$$

$$\frac{\partial}{\partial t}(\rho v) = -[\rho v \cdot \nabla v] - [v(\nabla \cdot \rho v)] - \nabla p - [\nabla \cdot \tau] + \rho g$$

$$\rho \frac{\partial v}{\partial t} + v \frac{\partial \rho}{\partial t} = \quad ''$$

$$\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v + v\left(\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho v)\right) = -\nabla p - [\nabla \cdot \tau] + \rho g$$

$$\boxed{\rho \frac{Dv}{dt} = -\nabla p - [\nabla \cdot \tau] + \rho g} \qquad 3.2\text{-}9$$

Navier-Stokes Eqn. constant ρ and μ.

$$\rho \frac{Dv}{dt} = -\nabla p + \mu \nabla^2 v + \rho g \quad \text{or} \quad -\nabla P + \mu \nabla^2 v$$

$$\nabla^2 = \text{laplacian} = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Stokes Flow Eqn. (creeping flow eqn.)

- acceleration is negligible $\rightarrow \rho \frac{Dv}{Dt} = 0$

$$0 = -\nabla p + \mu \nabla^2 v + \rho g$$

Euler Equation (viscous forces negligible).

$$\rho \frac{Dv}{Dt} = -\nabla p + \rho g.$$

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