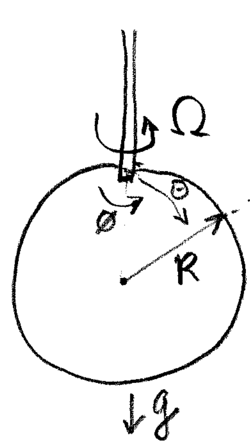
Flow Near a Slowly Rotating Sphere: Example 3.6-5



determine p, v, Torque

quiescent fluid away from sphere.

$v = \delta_\phi v_\phi(r,\theta)$ postulated!

$\mathcal{P} = \mathcal{P}(r,\theta)$

Eqn. of Continuity: all terms 0.

r-component of Eqn. of Motion: $0 = -\frac{\partial \mathcal{P}}{\partial r}$ $\quad -\frac{v_\phi^2}{r}$

θ-component : $0 = -\frac{1}{r}\frac{\partial \mathcal{P}}{\partial \theta}$

ϕ-component : $0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi \sin\theta)\right)$

BC1 $\quad r = R \quad v_r = 0,\ v_\theta = 0,\ v_\phi = R\Omega\sin\theta$

BC2 $\quad r \to \infty \quad v_r \to 0,\ v_\theta \to 0,\ v_\phi \to 0$

BC3 $\quad r \to \infty \quad \mathcal{P} \to p_0$ (far from sphere at $z=0$)

Try a solution $v_\phi = f(r)\sin\theta$ consistent with BC1!

Substitute into ϕ-component eqn.

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 2f = 0$$

Try $f = r^n$: subst. into above. $n = 1, -2$.

$$f = C_1 r + \frac{C_2}{r^2}$$

$$v_\phi = \left(C_1 r + \frac{C_2}{r^2}\right)\sin\theta$$

$C_1 = 0$, $C_2 = \Omega R^3$ when BCs are applied.

$$v_\phi = \Omega R \left(\frac{R}{r}\right)^2 \sin\theta$$

Torque = integral over sphere surface of tangential force, $(\tau_{r\phi}|_{r=R}) R^2 \sin\theta\, d\theta\, d\phi \cdot \underbrace{R\sin\theta}_{\text{lever arm for element.}}$

only component of $\tau \neq 0$!

$-\mu r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)$

$$\text{Torque} = \int_0^{2\pi}\int_0^{\pi} (\tau_{r\phi}|_{r=R}(R\sin\theta) R^2 \sin\theta\, d\theta\, d\phi$$ Table B.1

$$= \int_0^{2\pi}\int_0^{\pi} (3\mu\Omega\sin\theta)(R\sin\theta) R^2 \sin\theta\, d\theta\, d\phi$$

$$= 6\pi\mu\Omega R^3 \int_0^{\pi} \sin^3\theta\, d\theta$$

$$\boxed{= 8\pi\mu\Omega R^3}$$