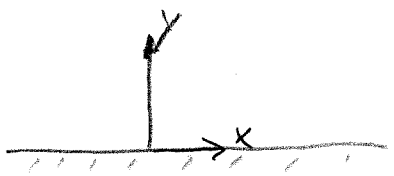
Chapter 4. v in terms of more than 1 Independent Variable

Time-Dependent Flows;

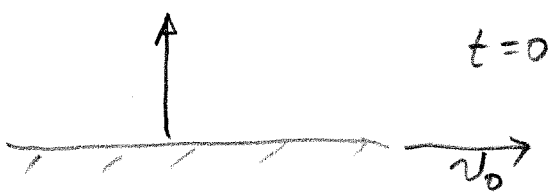
Flow Near a Boundary Set in Motion.

(Method of Similarity Solutions or Method of Combination of Variables).

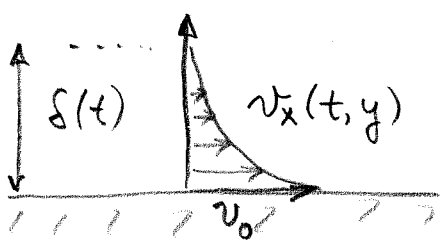


$t<0$, fluid at rest.

$\rho = \mu =$ constant



$t=0$, wall set in motion



$v_y = v_z = 0$, $p \neq p(x)$, $g_x = 0$

$t>0$, unsteady flow

Eqn. of Continuity, Table B.4.

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \cancel{\frac{\partial}{\partial x}(\rho v_x)}^{\,0} + \frac{\partial}{\partial y}(\rho \cancel{v_y}^{\,0}) + \frac{\partial}{\partial z}(\rho \cancel{v_z}^{\,0}) = 0$$

Eqn. of Motion, Table B.6

"kinematic viscosity" [cm²/s]

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2} \quad \text{or} \quad \frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}, \quad \nu \equiv \frac{\mu}{\rho}$$

Initial Condition (IC) : $t \le 0$, $v_x = 0$ for all y.

BC 1 : $y = 0$, $v_x = v_0$ for $t > 0$

BC 2 : $y = \infty$, $v_x = 0$ for $t > 0$.

Make Eqn. of Motion Dimensionless,

define $\phi \equiv v_x / v_0$ so $\dfrac{\partial v_x}{\partial t} = v_0 \dfrac{\partial \phi}{\partial t}$

$$\frac{\partial^2 v_x}{\partial y^2} = v_0 \frac{\partial^2 \phi}{\partial y^2}.$$

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \Rightarrow \frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2}.$$

$\phi = \phi(y, t; \nu)$, but is dimensionless, $\therefore$ it must be dependent on a dimensionless quantity made up of y, t, and ν.

$$\eta = \frac{y}{\sqrt{4\nu t}} \Rightarrow \phi = \phi(\eta)$$

We need to convert Eqn. of Motion and IC / BCs to dimensionless form.

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial \eta} \frac{\partial}{\partial t}\left(\frac{y}{2\nu^{1/2} t^{1/2}}\right) = \frac{\partial \phi}{\partial \eta} \frac{y}{4\nu^{1/2}} \left(-\frac{1}{2}\right) \frac{1}{t^{3/2}}$$

$$= \frac{\partial \phi}{\partial \eta} \left(-\frac{1}{2}\right) \frac{\eta}{t}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial \eta}\frac{1}{\sqrt{4\nu t}}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y}\left(\frac{\partial \phi}{\partial \eta}\frac{1}{\sqrt{4\nu t}}\right) = \frac{\partial^2 \phi}{\partial \eta^2}\frac{1}{4\nu t}$$

subst. into Eqn. of Motion

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2} \Rightarrow \frac{\partial \phi}{\partial \eta}\left(-\frac{1}{2}\right)\frac{\eta}{t} = \nu \frac{1}{4\nu t}\frac{\partial^2 \phi}{\partial \eta^2}$$

$$-\frac{\partial \phi}{\partial \eta}\eta = \frac{2\nu t}{4\nu t}\frac{\partial^2 \phi}{\partial \eta^2} \Rightarrow \frac{d^2\phi}{d\eta^2} + 2\eta\frac{d\phi}{d\eta} = 0$$

Through this combination of variables substitution, we have converted the Eqn. of Motion from a partial differential eqn. to an ordinary differential eqn. We need 2 "boundary" conditions to solve this eqn.

IC & BC2 $\rightarrow$ $\eta = \frac{y}{\sqrt{4\nu t}} = \frac{\infty}{\sqrt{4\nu t}} = \frac{y}{\sqrt{4\nu 0}} = \infty$

(BC.2 — — — $\phi = \frac{v_x}{v_0} = \frac{0}{v_0} = 0$

BC.1 — — — $\eta = \frac{0}{\sqrt{4\nu t}} = 0$

$\phi = v_x / v_0 = v_0 / v_0 = 1$

1/27/06