

# CBE 346 Spring 2013

## Princeton University

### Brief Overview of Process Control

Guest Lecturer: Dr. Tomas Co<sup>1</sup>

---

<sup>1</sup> Department of Chemical Engineering, Michigan Technological University  
Email: [tbco@mtu.edu](mailto:tbco@mtu.edu)

---

©2013 by Tomas Co

Page 1

### Brief Overview of Process Control

1. Elements of Process Control
2. Feedback Control
3. Dynamic Modeling
4. PID Controller Tuning
5. Analysis
6. Other Control Issues

## Process Control :

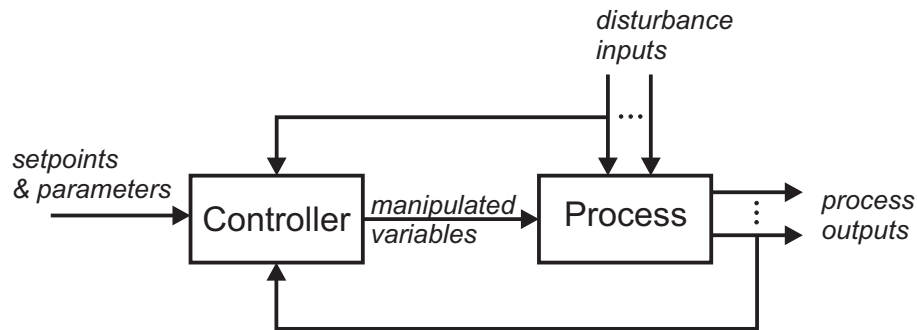
“... a statistics and engineering discipline that deals with architectures, mechanisms and algorithms for maintaining the output of a specific process within a desired range.”

- definition from wikipedia.org

## Elements of Process Control

1. Control objectives
  - *Setpoints (targets), constraints, specifications*
2. Input variables
  - *manipulated variable vs. disturbance variable*
3. Output variables
  - *controlled variable vs. uncontrolled variable*
  - *measured variable vs. unmeasured variable*
4. Control strategy
  - *Structure : feedback, feedforward ...*
  - *Control algorithms : On/Off, PID ...*

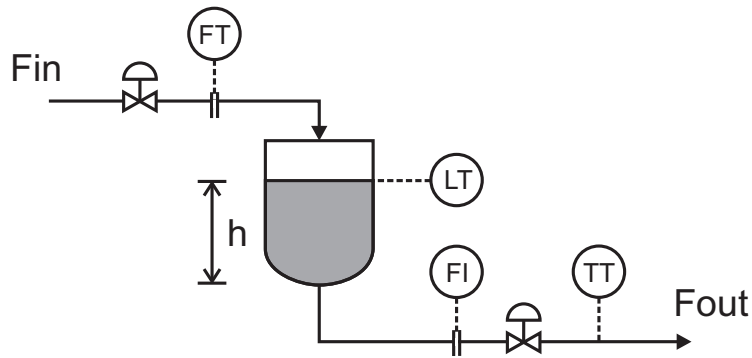
# Signal Flow Diagram



## Remarks:

1. Some control problems can be improved/simplified with design retrofits.
2. "Input" or "Output" refers to information flow - not material flow.
  - Two types of diagrams used in control design:
    - a. Signal block diagrams
    - b. Piping and instrumentation diagram (P&ID)

## Example 1: Level Control



**Objective:** Control the liquid level of a surge tank, where outlet is under gravity flow.

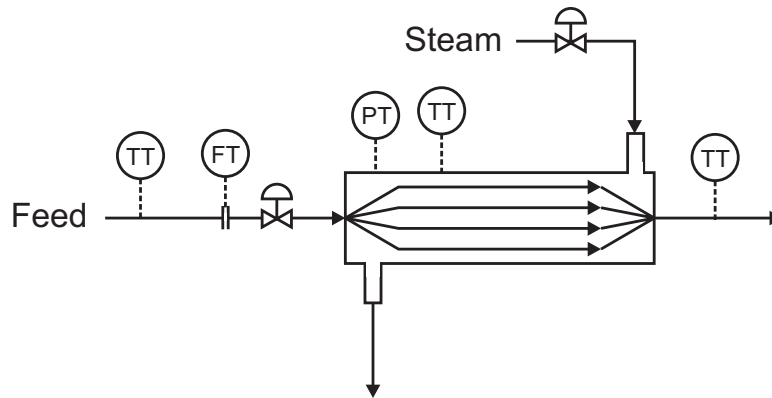
(**Sensors:** FT is flow transmitter, FI is flow indicator, LT is level transmitter, TT is temperature transmitter.)

(see <http://www.chem.mtu.edu/~tbco/cm416/pidiag.html>)

### Questions/Discussion:

1. Identify and classify the different variables.
2. Propose strategies to control the liquid level.
3. How tightly should the level be controlled?
4. Which control valves should be manipulated?

## Example 2: Heat Exchanger



### Questions/Discussion:

1. What is the control objective?
2. Identify and classify the different variables.
3. Propose control strategies.
4. If the product cannot exceed a maximum temperature, how does this affect the control strategy?

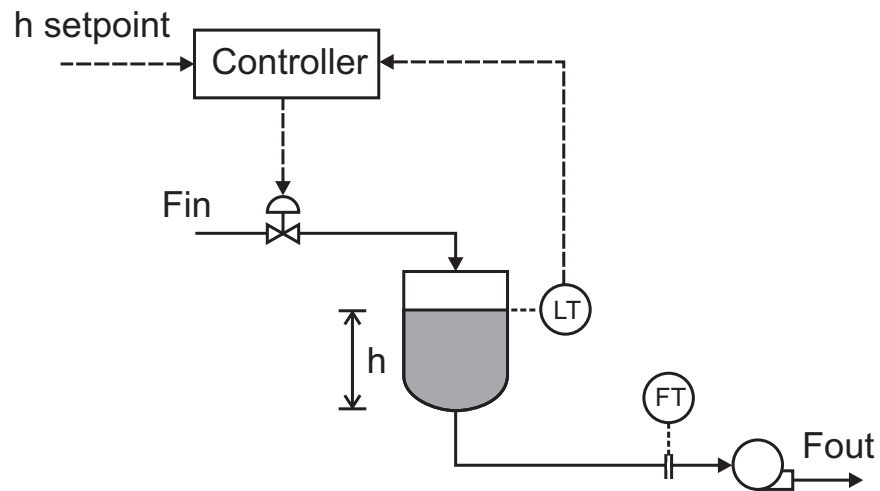
## Feedforward Control

Use input variables (e.g. disturbance measurements) to determine value of manipulated variable.

## Feedback Control

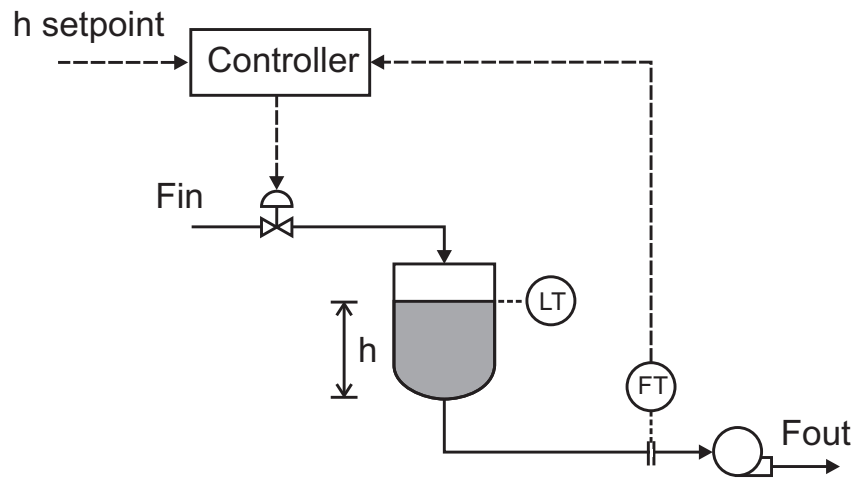
Use output variables (e.g. controlled variable) to determine value of manipulated variable.

### Case 1:



**Question:** Feedback or feedforward?

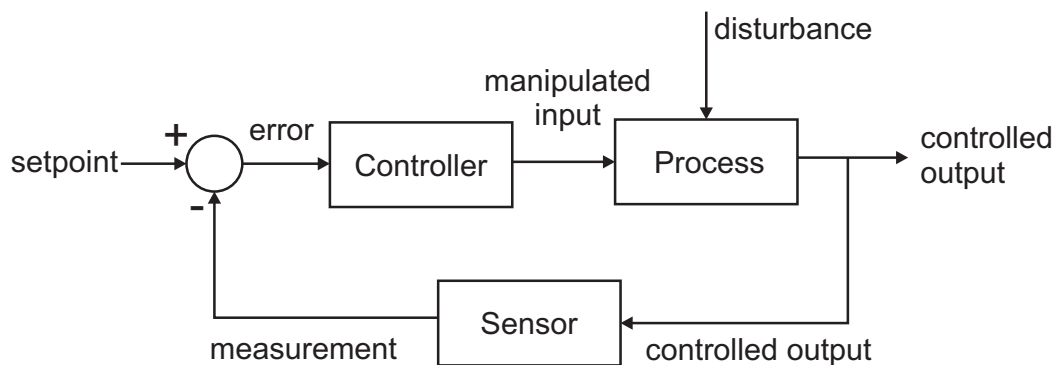
## Case 2:



**Questions:** Feedback or feedforward?

## Simple Feedback Control Structure:

(signal block diagram)



## General Roles of Feedback Control:

- Setpoint (target) tracking
- Disturbance rejection

## Relay (On/Off) Control: (ex.: home furnace, refrigerators)

$$u = \begin{cases} a & \text{if } y > y_{set} \\ b & \text{if } y \leq y_{set} \end{cases}$$

- Easiest (often cheapest) to implement
- Results in limit cycle response (often complemented with hysteresis to reduce erratic behavior due to measurement noise).
- Often:  $a = u_0 + h$  ;  $b = u_0 - h$  ( $h, u_0$  can be  $\leq 0$  or  $\geq 0$ )

## Exercise 1a: Manual Control and Relay Control

(<http://www.chem.mtu.edu/~tbco/cm416/newpida.html>)

1. Move setpoint to 0.6 then vary input ( $u$ ) to get output ( $y$ ) to approach setpoint value.
2. Switch to relay control and try with  $u_0 = 0$  and  $h = 0.1$ .  
(This mean  $a = -0.1$  and  $b = 0.1$ )
3. Try to improve the process with different values of  $u_0$  and  $h$ .
4. Move the setpoint to -1.



## OBSERVATIONS:

- 1.
- 2.
- 3.
- 4.

## Proportional Control Law:

$$u = K_c(y_{set} - y)$$

Where,  $K_c$  is known as the “Proportional Control Gain”.

**(Note:** often the algorithm includes a  $u_0$  term, called the “bias”.  
For simplicity, we will assume  $u_0 = 0$ .)

## Exercise 1b: Proportional Control

(<http://www.chem.mtu.edu/~tbco/cm416/newpida.html>)

1. Change setpoint to -0.2 then switch to Proportional control (PID mode with I and D mode switched off).
2. Try different values of proportional gain.

### OBSERVATIONS:

- 1.
- 2.
- 3.
- 4.

## Proportional-Integral Control ( to remove offsets )

$$u = K_c \left[ (y_{set} - y) + \frac{1}{\tau_I} \int (y_{set} - y) dt \right]$$

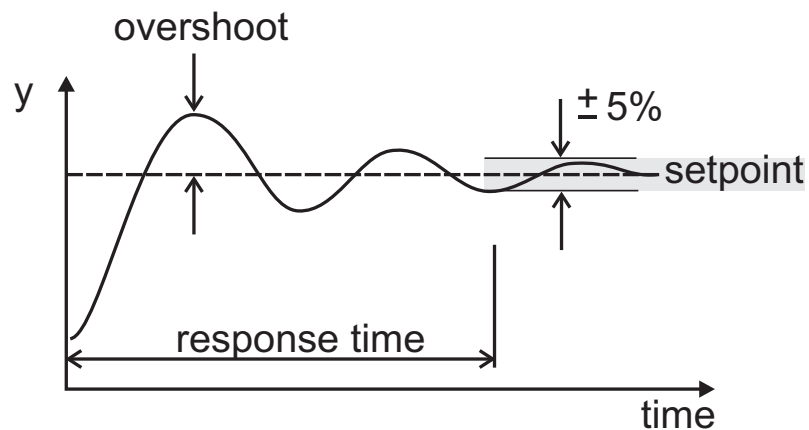
where,  $\tau_I$  = integral time constant, aka “reset time”.

- Simplified interpretation of  $\tau_I$ : projected average time for removing offset.
- Larger value of  $\tau_I$  reduces effects of integral mode.
- Smaller value of  $\tau_I$  likely to introduce “overshoot”.

### Exercise 1c: PI Control

(<http://www.chem.mtu.edu/~tbco/cm416/newpida.html>)

1. Move setpoint to 0.6 then set  $K_c = 0.2$  and  $\tau_I = 30$ .



2. Try other values to reduce overshoot.
3. Try other values to improve response time.

## OBSERVATIONS:

1.

2.

3.

4.

**Proportional-Integral-Derivative Control** ( to reduce oscillation/overshoot effects of integral mode )

$$u = K_c \left[ (y_{set} - y) + \frac{1}{\tau_I} \int (y_{set} - y) dt + \tau_d \frac{d(y_{set} - y)}{dt} \right]$$

where,  $\tau_d$ =derivative time (aka rate coefficient)

- Can improve (decrease) response time.
- Large value of  $\tau_d$  amplifies noise effects.

## Exercise 1d: PID Control

(<http://www.chem.mtu.edu/~tbco/cm416/newpida.html>)

1. Set  $K_c = 0.3$ ,  $\tau_I = 25$  and  $\tau_d = 8$ .
2. Try  $\tau_d = 50$ .

### OBSERVATIONS:

- 1.
- 2.
- 3.
- 4.

## Remarks:

1. P control is the simplest – often used for systems where offset is not a problem.
  - *Example: Level control of surge tanks*
2. PI control is used where offset is undesirable, yet responses to manipulated variables are fast.
  - *Example: Flow control*
3. PID control is used where offset is undesirable but responses are slow.
  - *Example: Temperature control*
4. Controller Tuning Problem: determining appropriate values of  $K_c$ ,  $\tau_I$  and  $\tau_d$ .

## RECAP #1

1. Four main elements of control :
  - i. Control objective
  - ii. Input variables
  - iii. Output variables
  - iv. Control strategy
2. Two main roles of control:
  - i. Setpoint tracking
  - ii. Disturbance rejection

### 3. Three modes of PID Control:

$$u = K_c \left[ (y_{set} - y) + \frac{1}{\tau_I} \int (y_{set} - y) dt + \tau_d \frac{d(y_{set} - y)}{dt} \right]$$

- i. Proportional Control Gain :  $K_c$
- ii. Integral-Time (Reset) :  $\tau_I$
- iii. Derivative-Time (Rate coefficient):  $\tau_d$

## Brief Overview of Process Control

1. Elements of Process Control
2. Feedback Control
3. Dynamic Modeling
4. PID Controller Tuning
5. Analysis
6. Other Control Issues

# Dynamic Process Models

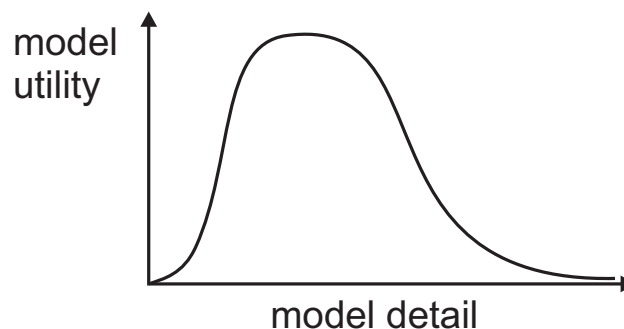
- Models used to
  - i) describe and simulate transient process behavior
  - ii) predict responses to different conditions
  - iii) explore effects of redesign/retrofits and/or control strategies
- Mathematical models: standard formulation involves differential equations based on time derivatives.

**Example:** heat exchanger

$$\frac{dT}{dt} = f \left( T, T_{in}, F, T_{steam}, P_{steam}, F_{steam}; U, A, c_p, \rho \right)$$

## Prater's Principle of Optimum Sloppiness

- There is an optimum level of model detail to yield maximum engineering utility based on the proposed objectives of the model (balanced among accuracy, cost, flexibility, etc.)



*(... but the optimum may change depending on availability and cost of new technologies.)*



-Types of models:

- a) Phenomenological (based on first principles)
- b) Empirical
- c) Mixed

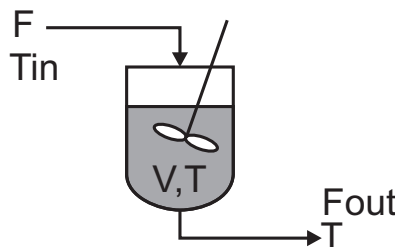
### Typical empirical models used in process control design:

- a) First order and first order plus time delay (FOPTD)
- b) Second order underdamped models
- c) Inverse-response models

### First Order Process:

$$\frac{dx}{dt} = \frac{1}{\tau} (K_p u - x)$$

### Example: Temperature in Continuous-Stirred Tank



$$\frac{d}{dt} (\rho V c_p [T - T_{ref}]) = \rho F c_p [T_{in} - T_{ref}] - \rho F_{out} c_p [T - T_{ref}]$$

Assume  $F = F_{out}$  and  $V, \rho, c_p$  constant:

$$\rightarrow \frac{dT}{dt} = \frac{F}{V} (T_{in} - T)$$

**Solution:** (use variation-of-parameters)

$$x(t) = x_0 e^{-t/\tau} + \int_0^t \frac{K_p u(\sigma)}{\tau} e^{\frac{[\sigma-t]}{\tau}} d\sigma$$

**Special Case:**  $u = u_{\text{new}}$  (constant),  $K_p u_0 = x_0$

$$\begin{aligned}x(t) &= x_0 e^{-t/\tau} + \frac{K_p u_{\text{new}}}{\tau} e^{-t/\tau} \int_0^t e^{\sigma/\tau} d\sigma \\&= x_0 e^{-t/\tau} + K_p u_{\text{new}} (1 - e^{-t/\tau}) \\&= x_0 + \left[ K_p u_{\text{new}} - x_0 \right] (1 - e^{-t/\tau}) \\&= x_0 + K_p [u_{\text{new}} - u_0] (1 - e^{-t/\tau})\end{aligned}$$

$$\frac{x(t) - x_0}{u_{\text{new}} - u_0} = K_p (1 - e^{-t/\tau})$$

## REMARKS:

1. As  $t \rightarrow \infty$ ,  $x \rightarrow x_{\text{new}}$ , a new steady state.
2. Then  $K_p = \Delta x_{SS} / \Delta u_{SS}$ , known as the “Process gain”, where  $\Delta x_{SS} = x_{\text{new}} - x_0$  and  $\Delta u_{SS} = u_{\text{new}} - u_0$ .
3. When  $t = \tau$ ,

$$\frac{x(\tau) - x_0}{x_{\text{new}} - x_0} = (1 - e^{-1}) = 0.632$$

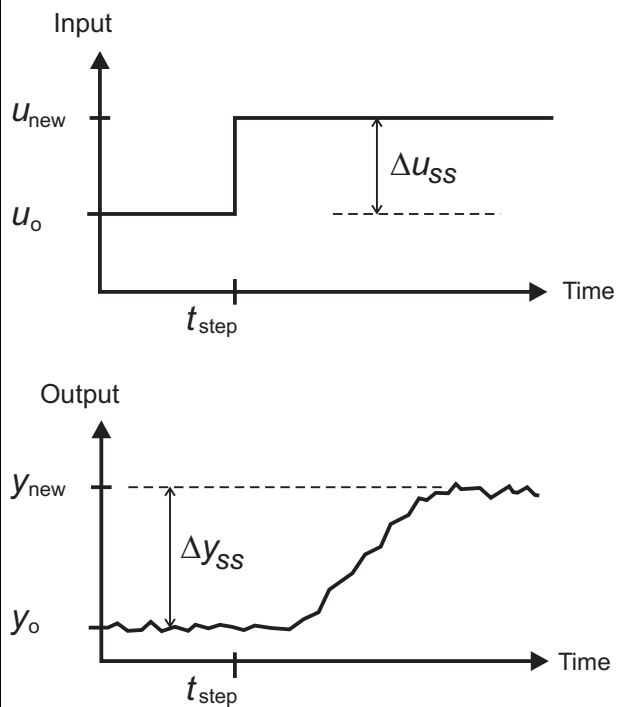
Thus,  $\tau$  is known as the “time constant” of the process (or the “0.632 point”).

**Question:** How does “half-life” compare to “time-constant”?

## Step-response experiment:

1. Fix manipulated input variable to  $u_o$  and wait until output settles to steady state  $y_o$ .
2. Introduce a step change in input to  $u_{new}$ . (Note the time when the step was implemented.)
3. Record the response of output  $y$  until it reaches a new equilibrium  $y_{new}$ .

$$\text{Process Gain}(K_p) = \frac{\Delta y_{ss}}{\Delta u_{ss}}$$



## First Order Plus Time-Delay (FOPTD) Model

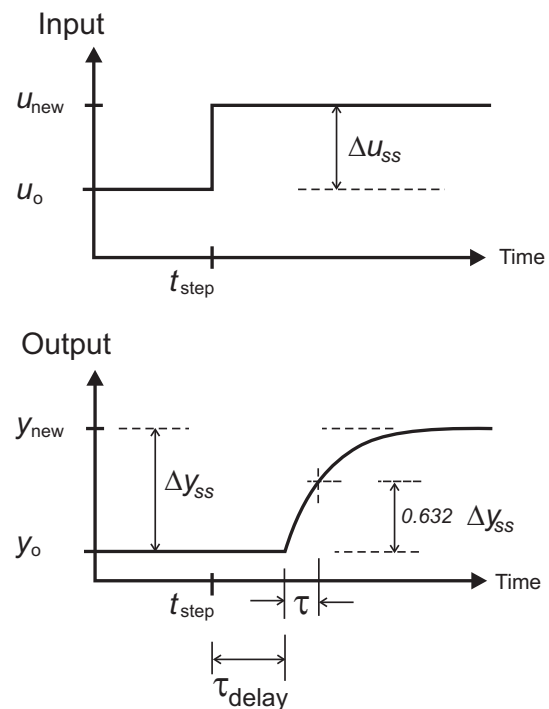
$$\frac{dy}{dt} = \frac{1}{\tau} [K_p u_{(t-\tau_{delay})} - y]$$

where,

$K_p$  = process gain

$\tau$  = time constant

$\tau_{delay}$  = time delay



## Analytical solution of FOPTD model subject to step test:

$$y(t) = \begin{cases} y_o & \text{if } t < t_{step} + \tau_{delay} \\ y_o + [1 - e^{-\beta(t)}]K_p\Delta u_{ss} & \text{if } t > t_{step} + \tau_{delay} \end{cases}$$

$$\beta(t) = \frac{t - t_{step} - \tau_{delay}}{\tau} \quad ; \quad \Delta u = u_{new} - u_o$$

## Estimation of $\tau$ , $K_p$ and $\tau_{delay}$

### Method 1:

$$K_p = \frac{\Delta y_{ss}}{\Delta u_{ss}} \quad (\text{by setting } t \rightarrow \infty)$$

Let  $t_a$  be the time such that  $\beta(t_a) = 1/3$ , then

$$\begin{aligned} y(t_a) &= y_o + K_p\Delta u_{ss}(1 - e^{-1/3}) \\ &= y_o + 0.283\Delta y_{ss} \end{aligned}$$

Let  $t_b$  be the time such that  $\beta(t_b) = 1$ , then

$$\begin{aligned} y(t_b) &= y_o + K_p\Delta u_{ss}(1 - e^{-1}) \\ &= y_o + 0.632\Delta y_{ss} \end{aligned}$$

From the experimental output, determine  $t_{step}$ ,  $t_a$ ,  $t_b$ .

$$t_a - t_{step} - \tau_{delay} = \frac{1}{3}\tau$$

$$t_b - t_{step} - \tau_{delay} = \tau$$

$$\rightarrow \tau = \frac{3}{2}(t_b - t_a) \quad ; \quad \tau_{delay} = t_b - t_{step} - \tau$$

**Method 2:** use computers (e.g. MS Excel)

## Exercise 2: Parameter Estimation of FOPTD

(<http://www.chem.mtu.edu/~tbco/cm416/newpida.html>)

1. Implement a step test.
2. Collect a range of data that contains initial steady state and final steady state. Estimate the model parameters using method 1.
3. Use MS Excel to estimate model parameters using the analytical solution of FOPTD.

## Cohen-Coon PID Tuning Rules:

Based on FOPTD, obtain  $K_p$ ,  $\tau$  and  $\tau_{delay}$ .

$$\text{Let } r = \frac{\tau_{delay}}{\tau},$$

	$K_c$	$\tau_I$	$\tau_d$
$P$	$\frac{1}{K_p r} \left(1 + \frac{r}{3}\right)$		
$PI$	$\frac{1}{K_p r} \left(0.9 + \frac{r}{12}\right)$	$\tau_{delay} \frac{30 + 3r}{9 + 20r}$	
$PID$	$\frac{1}{K_p r} \left(\frac{4}{3} + \frac{r}{4}\right)$	$\tau_{delay} \frac{32 + 6r}{13 + 8r}$	$\tau_{delay} \frac{4}{11 + 2r}$

### Exercise 3: Cohen-Coon Tuning

(<http://www.chem.mtu.edu/~tbco/cm416/newpida.html>)

1. Use FOPTD parameters to find PID parameters.
2. Implement PID.

# Closed-loop Modeling for Ziegler-Nichols Tuning

<ol style="list-style-type: none"> <li>1. Implement P Control.</li> <li>2. Obtain <b>ultimate gain</b>, <math>K_u</math> (the critical value of <math>K_c</math> where the process is about to be unstable.)</li> <li>3. At <math>K_c = K_u</math>, measure the <b>ultimate period</b> <math>P_u</math> (the time from one peak to the next).</li> </ol>	
--	--

## Ziegler-Nichols PID Tuning Rules:

Using  $K_u$  and  $P_u$ ,

	$K_c$	$\tau_I$	$\tau_d$
$P$	$K_u/2$		
$PI$	$K_u/2.2$	$P_u/1.2$	
$PID$	$K_u/1.7$	$P_u/2$	$P_u/8$

## Tyreus-Luyben PID Tuning Rules:

Using  $K_u$  and  $P_u$ ,

	$K_c$	$\tau_I$	$\tau_d$
$P$	$K_u/2$		
$PI$	$K_u/3.2$	$2.2P_u$	
$PID$	$K_u/2.2$	$2.2P_u$	$P_u/6.3$

### Exercise 4: Ziegler-Nichols Tuning

(<http://www.chem.mtu.edu/~tbco/cm416/newpida.html>)

1. Find  $K_u$  and  $P_u$ .
2. Evaluate PID parameters based on Ziegler-Nichols rules.
3. Implement PID.
4. Repeat with Tyreus-Luyben.



## Second-order Underdamped Processes:

$$\tau_n^2 \frac{d^2 y}{dt^2} + 2\tau_n \zeta \frac{dy}{dt} + y = K_p u$$

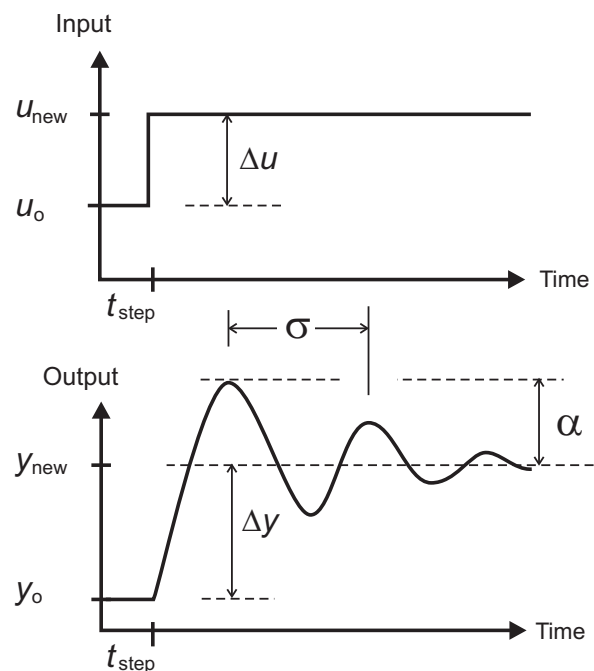
where,  $\zeta$  = damping coefficient

$\tau_n$  = natural time constant

$K_p$  = process gain =  $\Delta y_{SS} / \Delta u_{SS}$

$$\rightarrow \frac{\alpha}{\Delta y_{SS}} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

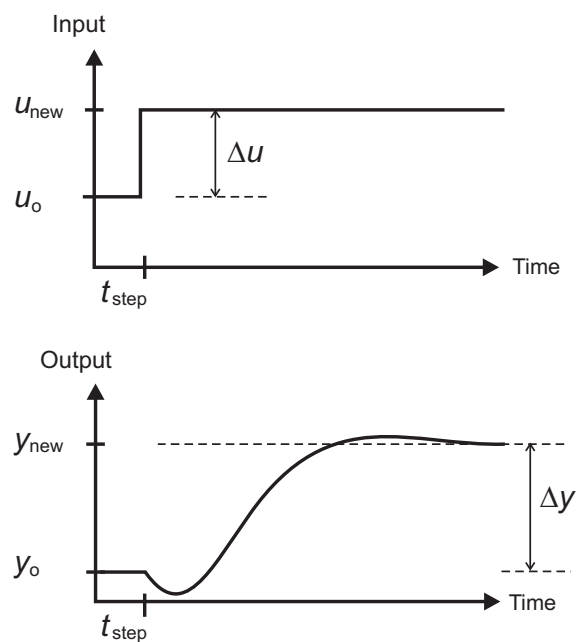
$$\sigma = \frac{2\pi\tau_n}{\sqrt{1-\zeta^2}}$$



## Inverse Response Processes:

$$\hat{a}_2 \frac{d^2 y}{dt^2} + \hat{a}_1 \frac{dy}{dt} + \hat{a}_0 y = K_p \left( -\hat{\beta}_1 \frac{du}{dt} + u \right)$$

-needs numerical methods to estimate parameters



## General 2<sup>nd</sup> Order Linear Model:

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0y = b_1 \frac{du}{dt} + b_0u$$

Is equivalent to

$$\begin{aligned}\frac{dz_1}{dt} &= -a_1z_1 + z_2 + b_1u \\ \frac{dz_2}{dt} &= -a_0z_1 + b_0u \\ y &= z_1\end{aligned}$$

## General n<sup>th</sup> Order Linear Model:

$$\frac{d^ny}{dt^n} + \dots + a_1 \frac{dy}{dt} + a_0y = b_{n-1} \frac{d^{n-1}u}{dt^{n-1}} + \dots + b_0u$$

Is equivalent to

$$\begin{aligned}\frac{dz_1}{dt} &= -a_{n-1}z_1 + z_2 + b_{n-1}u \\ &\vdots \\ \frac{dz_{n-1}}{dt} &= -a_1z_1 + z_n + b_1u \\ \frac{dz_n}{dt} &= -a_0z_1 + b_0u \\ y &= z_1\end{aligned}$$

## Computer Simulation to Estimate Parameters

Euler Method:

$$\frac{dy}{dt} = f(y, u, t) \approx \frac{y_{k+1} - y_k}{\Delta t} = f(y_k, u_k, t_k)$$

$$\rightarrow y_{k+1} = y_k + \Delta t f(y_k, u_k, t_k)$$

So for  $n^{th}$  order process,

$$\begin{aligned}(z_1)_{k+1} &= (z_1)_k + \Delta t[-a_{n-1}(z_1)_k + (z_2)_k + b_{n-1}(u)_k] \\ &\quad \vdots \\ (z_n)_{k+1} &= (z_n)_k + \Delta t[-a_0(z_1)_k + b_0(u)_k] \\ y_{k+1} &= (z_1)_{k+1}\end{aligned}$$

**Q:** What about initial conditions?

**A:** For convenience, it would be helpful if the initial conditions were all zero.

This can be accomplished if:

- The process is initially at equilibrium  $\rightarrow$  all time derivatives are zero
- The variables are replaced by “deviation variables”

$$y_{dev} = y - y_o \quad \text{and} \quad u_{dev} = u - u_o$$

**Note:** Using these tricks will also help later when building transfer functions.

For PID, let  $e_k = y_{set} - y_k$ ,

$$u_{k+1} = K_c \left( e_k + \frac{\Delta t}{\tau_I} \sum_{i=0}^k e_i + \frac{\tau_d}{\Delta t} [e_k - e_{k-1}] \right)$$

$$u_k = K_c \left( e_{k-1} + \frac{\Delta t}{\tau_I} \sum_{i=0}^{k-1} e_i + \frac{\tau_d}{\Delta t} [e_{k-1} - e_{k-2}] \right)$$

After subtraction, we get the “discrete PID” form:

$$u_{k+1} = u_k + K_c \left( e_k - e_{k-1} + \frac{\Delta t}{\tau_I} e_k + \frac{\tau_d}{\Delta t} [e_k - 2e_{k-1} + e_{k-2}] \right)$$

## Exercise 5: Optimal Tuning from Simulation

(<http://www.chem.mtu.edu/~tbco/cm416/newpidb.html>)

1. Obtain step test data.
2. Use MS Excel to approximate the model.
3. Use the model to find optimal tuning.
4. Implement the PID parameters.

## RECAP # 2

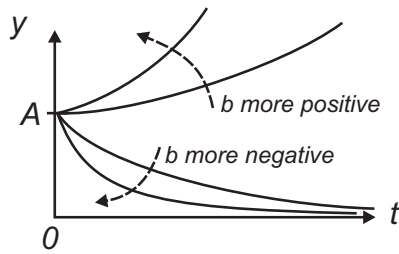
1. Models (at various levels of details) are used to help characterize the dynamics of a process.
2. If FOPTD applies, then Cohen-Coon tuning rules apply. Alternatively, the Ziegler-Nichols tuning is also often used for PID tuning.
3. Computer simulation can also be used to estimate the model and this can be used for optimal tuning of PID controllers.

## Brief Overview of Process Control

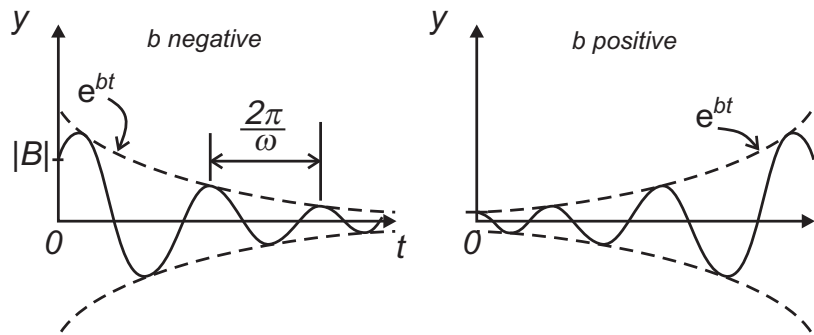
1. Elements of Process Control
2. Feedback Control
3. Dynamic Modeling
4. PID Controller Tuning
5. Analysis
6. Other Control Issues

# Typical Dynamic Elements

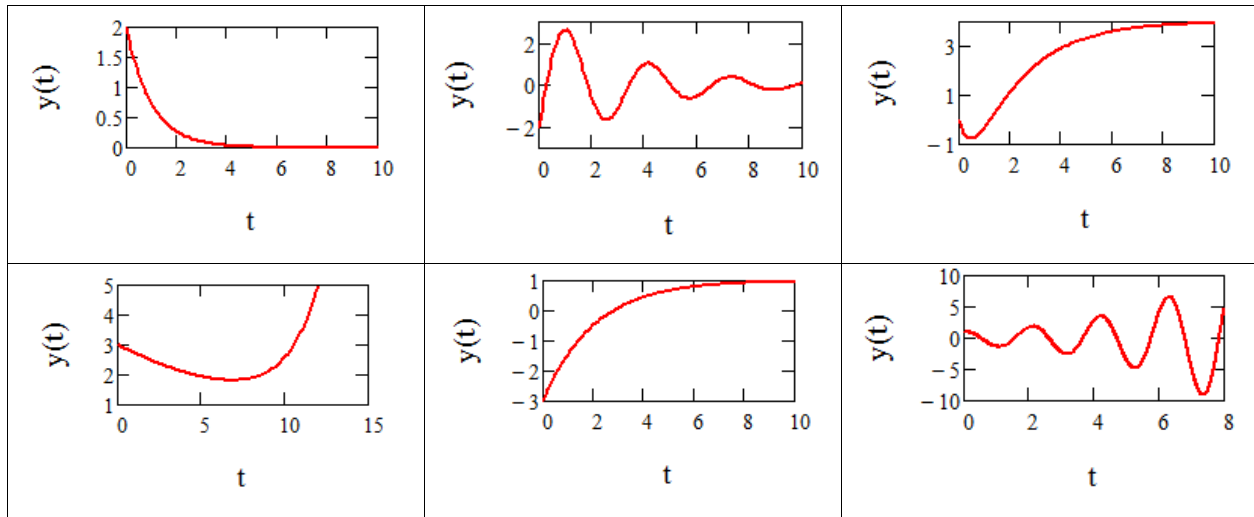
1. Exponential Decay or Growth:  $y(t) = Ae^{bt}$



2. Sinusoidal Response:  $y(t) = e^{bt}[A \sin(\omega t) + B \cos(\omega t)]$



Q: Which functions will match the graphs below?



a) $1 - 4e^{-t/2}$	b) $4 - 8e^{-t/2} + 4e^{-2t}$	c) $e^{-0.3t}[3 \sin(2t) - 2 \cos(2t)]$	d) $2e^{-0.1t}$
e) $e^{0.3t} \cos(3t)$	f) $2e^{-t}$	g) $e^{-0.2t}[\sin(20t) + 4 \cos(20t)]$	h) $0.01e^{0.5t} + 3e^{-0.1t}$

## Solution of ODE using Laplace Transforms

**Definition:** Given  $f(t)$ , then **Laplace transform** is given by

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt = \tilde{f}(s) \quad ; \quad \text{Re}(s) > 0.$$

**Example:**  $f(t) = e^{-at}$ , where  $a$  is a constant.

$$\begin{aligned}\mathcal{L}[e^{-at}] &= \int_0^{\infty} (e^{-at})e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt \\ &= -\frac{1}{s+a} e^{-(s+a)t} \Big|_{t=0}^{t=\infty} = \frac{1}{s+a}\end{aligned}$$

Special case:  $a = 0, \rightarrow \mathcal{L}[1] = 1/s.$

### Simple Laplace Transform table:

$f(t)$	$\tilde{f}(s) = \mathcal{L}[f]$
$e^{-at}$	$\frac{1}{s+a}$
$e^{-at} \cos(bt)$	$\frac{(s+a)}{(s+a+ib)(s+a-ib)}$
$e^{-at} \sin(bt)$	$\frac{b}{(s+a+ib)(s+a-ib)}$
$t^n$	$\frac{n!}{s^{n+1}}$

(where  $i = \sqrt{-1}$ )

## Laplace transform of derivatives:

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^{\infty} \left(\frac{df}{dt}\right) e^{-st} dt$$

Integration by parts:  $u = e^{-st} \quad ; \quad du = -se^{-st}$   
 $dv = \frac{df}{dt} dt \quad ; \quad v = f$

$$\begin{aligned} \rightarrow \mathcal{L}\left[\frac{df}{dt}\right] &= f e^{-st} \Big|_{t=0}^{t=\infty} + s \int_0^{\infty} f e^{-st} dt \\ &= -f(0) + s\mathcal{L}[f] \end{aligned}$$

## Generalizing:

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n \mathcal{L}[f] - \left( s^{n-1} f(0) + \sum_{k=1}^{n-1} s^k \frac{d^k f}{dt^k} \Big|_{t=0} \right)$$

**Linearity Property:** let  $a$  and  $b$  be constant, then

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f] + b\mathcal{L}[g]$$

## Inverse Laplace Transform:

$$\mathcal{L}^{-1}[\tilde{f}(s)] = \mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t)$$

- Often use table of Laplace transforms, if item is available
- If necessary, can use the Bromwich formula (*quite rarely*)

$$\mathcal{L}^{-1}[\tilde{f}(s)] = \frac{1}{2\pi i} \left[ \lim_{\sigma \rightarrow \infty} \int_{\gamma - \sigma}^{\gamma + \sigma} \tilde{f}(s) e^{st} ds \right]$$

## Example:

$$\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$



**Example:** Obtain the solution of ODE using Laplace transforms

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3 \quad ; \quad y(0) = 0; \quad \left. \frac{dy}{dt} \right|_{t=0} = 0$$

Apply Laplace transforms of both sides,

$$s^2\mathcal{L}[y] + 5s\mathcal{L}[y] + 6\mathcal{L}[y] = \frac{3}{s}$$

$$\mathcal{L}[s] = \frac{3}{s(s^2 + 5s + 6)} = \frac{3}{s(s + 3)(s + 2)} = \frac{A}{s} + \frac{B}{s + 3} + \frac{C}{s + 2}$$

$$A = 1/2, \quad B = 1, \quad C = -3/2$$

$$y(t) = A\mathcal{L}^{-1}\left[\frac{1}{s}\right] + B\mathcal{L}^{-1}\left[\frac{1}{s + 3}\right] + C\mathcal{L}^{-1}\left[\frac{1}{s + 2}\right]$$

$$= A + Be^{-3t} + Ce^{-2t}$$

**Example:** Obtain the solution of ODE using Laplace transforms

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 3 \quad ; \quad y(0) = 0; \quad \left. \frac{dy}{dt} \right|_{t=0} = 0$$

Apply Laplace transforms of both sides,

$$s^2\mathcal{L}[y] + 2s\mathcal{L}[y] + 5\mathcal{L}[y] = \frac{3}{s}$$

$$\mathcal{L}[y] = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{s(s + 1 + 2i)(s + 1 - 2i)}$$

$$= \frac{A}{s} + \frac{B}{(s + 1 + 2i)(s + 1 - 2i)} + \frac{C(s + 1)}{(s + 1 + 2i)(s + 1 - 2i)}$$

$$y(t) = A + Be^{-t} \sin(2t) + Ce^{-t} \cos(2t)$$

$$\text{where } A = 3/5, B = -3/10, C = 3/5$$

**For the general nth order linear model, assuming zero initial conditions:**

$$a_n \frac{d^n y}{dt^n} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_0 u$$

Taking the Laplace transforms yields

$$(a_n s^n + \dots + a_1 s + a_0) \tilde{y} = (b_{n-1} s^{n-1} + \dots + b_0) \tilde{u}$$

or

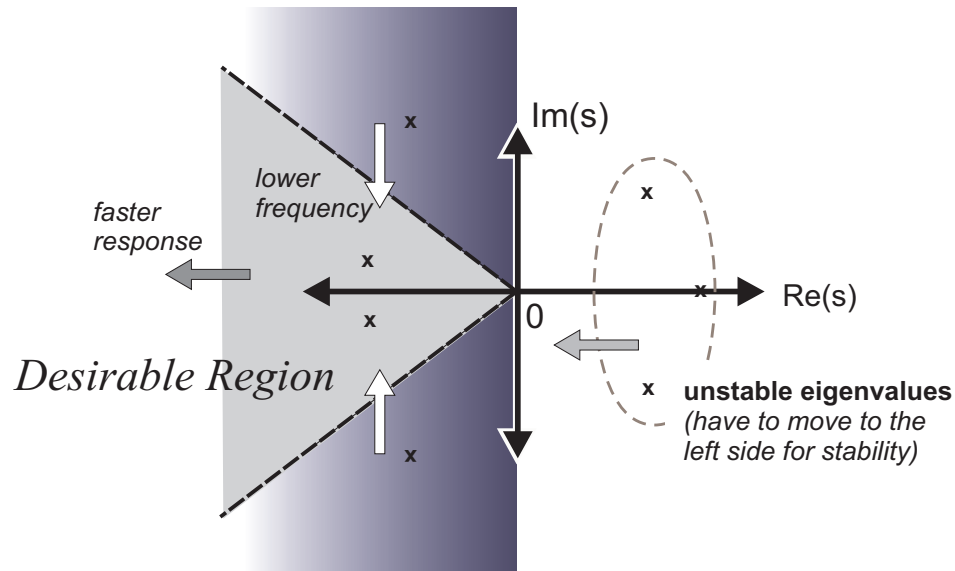
$$\tilde{y} = \left[ \frac{b_{n-1} s^{n-1} + \dots + b_0}{a_n s^n + \dots + a_1 s + a_0} \right] \tilde{u} = G(s) \tilde{u}$$

### **Remarks:**

1.  $G(s)$  is the transfer function from  $\tilde{u}$  to  $\tilde{y}$ .
2. The roots of the denominator are known as the “**poles**” of the transfer function, also known as the “**eigenvalues**” of the process.
3. The eigenvalues determine the transient behavior of the process:
  - a) If any of the eigenvalues have a positive real part, then the process will be unstable.
  - b) The more negative the real part, the faster the dynamics die out.
  - c) The imaginary parts of the eigenvalues determine the frequency of oscillations.

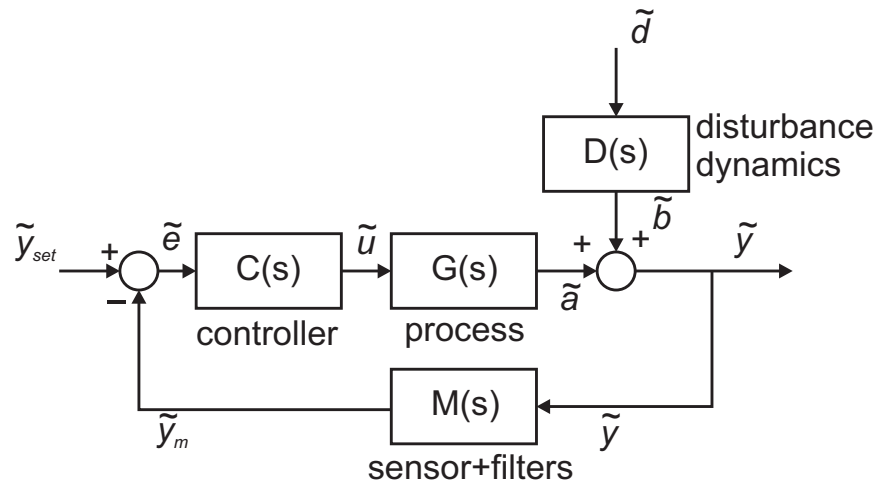
## Main Principle for Linear Control Design

Feedback controllers, compensators and control configurations are designed to alter the system dynamics by adjusting the values (i.e. position in complex plane) of the eigenvalues.



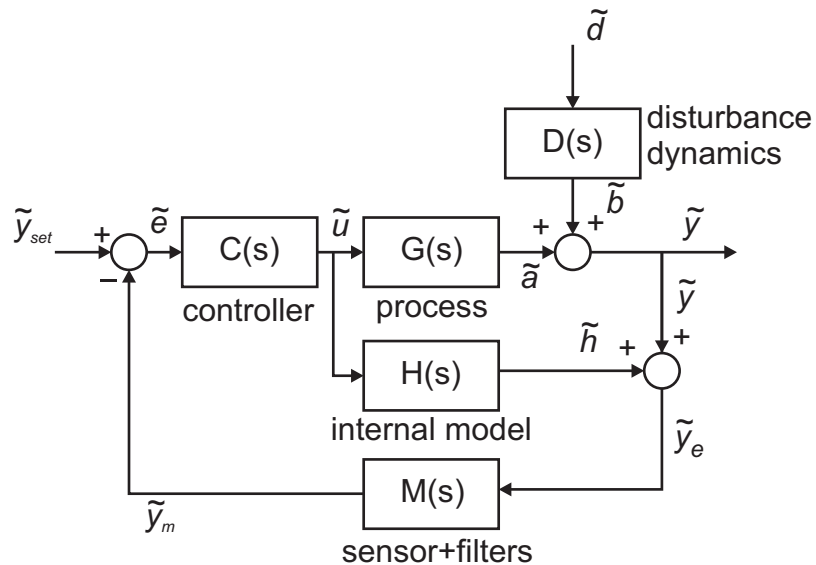
- ➔ By considering each block in a signal flow diagram to have a transfer function, the overall equivalent transfer function from the setpoint to the output can be found by simple algebraic manipulation.
- ➔ Likewise, the overall equivalent transfer function from disturbance to the output can also be found by algebraic manipulations.

## Example: Simple feedback control



$\begin{aligned} \tilde{y} &= \tilde{a} + \tilde{b} \\ \tilde{a} &= G\tilde{u} \\ \tilde{b} &= D\tilde{d} \\ \tilde{u} &= C\tilde{e} \\ \tilde{e} &= \tilde{y}_{set} - \tilde{y}_m \\ \tilde{y}_m &= M\tilde{y} \end{aligned}$	$\begin{aligned} \tilde{y} &= GC(\tilde{y}_{set} - M\tilde{y}) + D\tilde{d} \\ (1 + GCM)\tilde{y} &= GC\tilde{y}_{set} + D\tilde{d} \\ \tilde{y} &= \left[ \frac{GC}{1 + GCM} \right] \tilde{y}_{set} + \left[ \frac{D}{1 + GCM} \right] \tilde{d} \end{aligned}$
--	---

## Challenge: Internal model control



$$\tilde{y} = (?)\tilde{y}_{set} + (?)\tilde{d}$$

## Transfer Functions of P, PI and PID Controllers

P	$K_c$
PI	$K_c \frac{\tau_I s + 1}{\tau_I s}$
PID	$K_c \left( \frac{\tau_I s + 1}{\tau_I s} \right) \left( \frac{\tau_d s + 1}{\alpha \tau_d s + 1} \right) ; \alpha < 0.05$

### Example: Simple Feedback Control (continuation)

Let  $C(s) = K_c$ ,  $M(s) = 1$ ,  $D(s) = \frac{2}{s-2}$  and  $G(s) = -\frac{3}{s-2}$ .

Then, after substitution,

$$\begin{aligned} \tilde{y} &= \left( \frac{K_c \left( -\frac{3}{s-2} \right)}{1 + K_c \left( -\frac{3}{s-2} \right)} \right) \tilde{y}_{set} + \left( \frac{\frac{2}{s-2}}{1 + K_c \left( -\frac{3}{s-2} \right)} \right) \tilde{d} \\ &= \left( \frac{-3K_c}{s-2-3K_c} \right) \tilde{y}_{set} + \left( \frac{2}{s-2-3K_c} \right) \tilde{d} \end{aligned}$$

For stabilization, we need:  $K_c < -\frac{2}{3}$

**Q:** What about estimation of error offset ?

**A:** One can use the final value theorem of Laplace transform.

---

**Final value theorem:** Assuming  $f(t)$  is stable,

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \mathcal{L}[f]$$

**Proof:** from Laplace transform of derivative

$$\begin{aligned} \lim_{s \rightarrow 0} s \mathcal{L}[f] &= \lim_{s \rightarrow 0} \int_0^{\infty} \frac{df}{dt} e^{-st} dt + f(0) = \int_{f(0)}^{f(\infty)} df + f(0) \\ &= f(\infty) \end{aligned}$$

**Example:** (continuation from previous example)

$$\tilde{y} = \left( \frac{-3K_c}{s-2-3K_c} \right) \tilde{y}_{set} + \left( \frac{2}{s-2-3K_c} \right) \tilde{d}$$

Assume  $y_{set} = 3$  and  $d = 2$ .

$$\tilde{e}(s) = \tilde{y}_{set} - \tilde{y} = \left( \frac{3}{s} \right) - \left( \frac{-3K_c}{s-2-3K_c} \right) \left( \frac{3}{s} \right) - \left( \frac{2}{s-2-3K_c} \right) \left( \frac{2}{s} \right)$$

The offset is then given by

$$\begin{aligned} e(\infty) &= \lim_{s \rightarrow 0} s \left[ \left( \frac{3}{s} \right) - \left( \frac{-3K_c}{s-2-3K_c} \right) \left( \frac{3}{s} \right) - \left( \frac{2}{s-2-3K_c} \right) \left( \frac{2}{s} \right) \right] \\ &= \frac{3(-2-3K_c) + 9K_c - 4}{-2-3K_c} = \frac{10}{3K_c + 2} \end{aligned}$$

→ as  $K_c \ll -\frac{2}{3}$ , the smaller the offset.

**Q:** Laplace transforms and transfer functions are only valid for linear dynamics. What about nonlinear systems?

**A:** If process are expected to be operating in a small region around a set of nominal values, then linearization can be used, i.e. the eigenvalue analysis will be valid (around the small region).

**Note:** one particular feature of nonlinear systems is the possibility of multiple steady states

**Linearization around an operating point  $(u_o, y_o, d_o)$ :**

$$\begin{aligned}\frac{dy}{dt} &= f(y, u, d) \\ &\approx a(y - y_o) + b(u - u_o) + c(d - d_o) + q\end{aligned}$$

where,  $q = f(y_o, u_o, d_o)$  ;  $a = \left. \frac{\partial f}{\partial y} \right|_{y_o, u_o, d_o}$

$$b = \left. \frac{\partial f}{\partial u} \right|_{y_o, u_o, d_o} ; \quad c = \left. \frac{\partial f}{\partial d} \right|_{y_o, u_o, d_o}$$

**Common simplification:** Use deviation variables,  $y_{dev} = (y - y_o), \dots$ , and assuming  $(y_o, u_o, d_o)$  is at equilibrium,

$$\frac{dy_{dev}}{dt} \approx ay_{dev} + bu_{dev} + cd_{dev}$$

### Example:

$$\frac{dT}{dt} = T^2 - 80T + 10u^2 + 250d = f(T, u, d)$$

At operating point 1:  $(T_o, u, d) = (30, 10, 2)$

$$\left. \frac{\partial f}{\partial T} \right|_{(T=30, u=10, d=2)} = 2T_o - 80 = -20$$

$$\left. \frac{\partial f}{\partial u} \right|_{(T=30, u=10, d=2)} = 200 \quad ; \quad \left. \frac{\partial f}{\partial d} \right|_{(T=30, u=10, d=2)} = 250$$

$$\frac{dT_{dev}}{dt} = -20T_{dev} + 200u_{dev} + 250d_{dev} \quad (\text{stable})$$

At operating point 2:  $(T_o, u, d) = (50, 10, 2)$

$$\frac{dT_{dev}}{dt} = +20T_{dev} + 200u_{dev} + 250d_{dev} \quad (\text{unstable})$$

### RECAP #3

1. The eigenvalues are key tools for analysis of the dynamics with or without controllers.
  - If any of the eigenvalues has positive real parts, the system will be unstable
  - The more negative the real parts the faster the response
  - The larger the imaginary parts, the higher the frequency of oscillation
2. Using Laplace transforms, we can characterize the effects of inputs to the outputs via transfer functions.



3. Only algebraic manipulations are needed to obtain the transfer functions from either setpoint or disturbance to the process output.
4. Control design, configuration and tuning is focused on how to move the eigenvalues to locations in the complex plane that would achieve desired dynamic behavior.
5. If system is nonlinear, linear analysis can be used on linearized approximate models.

## Other Issues in Classical Process Controls

1. Signal filtering
  - *Need to smooth out noise without damping crucial dynamic information*
2. Anti-reset windup
  - *Integral mode accumulate error information even though valves/control-elements have saturated, causing unnecessary inertial effects on controller response.*
3. Cascade control
  - *Direct feedback control be become sluggish due to nonlinearities (e.g. valve stiction).*
4. Split-range control
  - *Control elements are often directional, e.g. cooling and heating elements have different dynamic effects.*

## 5. Robustness and Auto-tuning

- *Require new controller parameters when set-points or process dynamics are significantly far from nominal design conditions*

## 6. Multivariable and plant-wide control

- *Various control configuration are possible: cascade, multiple single-input/single-output (SISO) control loops, multi-input/multi-output (MIMO) control loops, etc.*

### **Other Control Strategies:**

1. Cascade Control
2. Feedforward-Feedback Control
3. Internal Model Control (special case: Smith predictor)
4. Model Predictive Control

## Model Predictive Control:

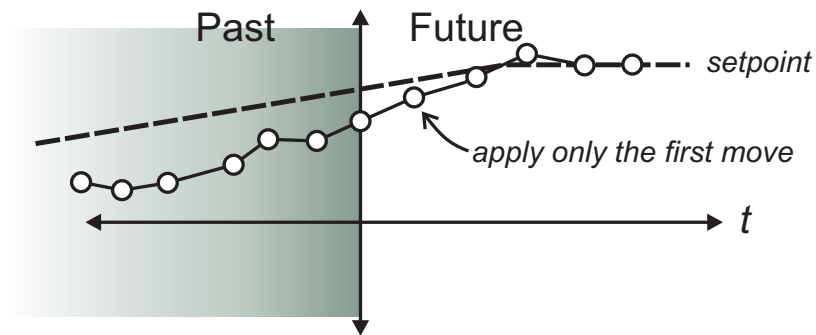
1. Use optimization to evaluate N-steps ahead:

$$\min_{u_1, \dots, u_N} \text{Cost}(y, y_{set}, u)$$

Subject to:  $\dot{y} = f(y, u, d)$

$$y_{\min} \leq y \leq y_{\max}; u_{\min} \leq u \leq u_{\max}$$

2. Implement only one step (or a few steps)
3. Repeat from step 1.



## OVERALL RECAP

1. Introduction to control concepts
  - elements and feedback control
2. PID control and tuning rules
  - Control law: P, PI and PID
  - Ziegler-Nichols and Cohen-Coon
  - Optimal tuning approach
3. Process modeling
  - FOPTD model
  - General linear model
  - Simulation and parameter estimation

#### 4. Analysis

- Using eigenvalues to predict behavior
- Laplace transforms to generate transfer functions
- Analysis and design of feedback system using transfer function manipulation
- Linearization

#### 5. Other control issues and advanced control configurations.