# The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

**CM3110 Fall 2011 Faith A. Morrison**

**Continuity Equation**, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

**Continuity Equation**, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

**Continuity Equation**, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

**Equation of Motion** for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned}
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left( \frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x \\
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left( \frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y \\
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left( \frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z
\end{aligned}$$

**Equation of Motion** for an incompressible fluid, 3 components in cylindrical coordinates

$$\begin{aligned}
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \left( \frac{1}{r}\frac{\partial (r\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z} \right) + \rho g_r \\
\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left( \frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} \right) + \rho g_\theta \\
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left( \frac{1}{r}\frac{\partial (r\tilde{\tau}_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z
\end{aligned}$$

**Equation of Motion** for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right)$$

$$= -\frac{\partial P}{\partial r} + \left( \frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r$$

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r} \right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left( \frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi}\cot\theta}{r} \right) + \rho g_\theta$$

$$\rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot\theta}{r} \right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \left( \frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \left(\tilde{\tau}_{\theta\phi}\sin\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta}\cot\theta}{r} \right) + \rho g_\phi$$

**Equation of Motion** for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

**Equation of Motion** for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

**Equation of Motion** for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right.$$
$$\left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right.$$
$$\left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi$$

Note: the $r$-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.


References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, $2^{nd}$ edition, Wiley: NY, 2002.

2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

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# FACTORS FOR UNIT CONVERSIONS

| Quantity | Equivalent Values |
|---|---|
| Mass | 1 kg = 1000 g = 0.001 metric ton = 2.20462 $lb_m$ = 35.27392 oz |
| | 1 $lb_m$ = 16 oz = 5 x $10^{-4}$ ton = 453.593 g = 0.453593 kg |
| Length | 1 m = 100 cm = 1000 mm = $10^6$ microns ($\mu$m) = $10^{10}$ angstroms (Å) |
| | = 39.37 in = 3.2808 ft = 1.0936 yd = 0.0006214 mile |
| | 1 ft = 12 in. = 1/3 yd = 0.3048 m = 30.48 cm |
| Volume | 1 $m^3$ = 1000 liters = $10^6$ $cm^3$ = $10^6$ ml |
| | = 35.3145 $ft^3$ = 220.83 imperial gallons = 264.17 gal |
| | = 1056.68 qt |
| | 1 $ft^3$ = 1728 $in^3$ = 7.4805 gal = 0.028317 $m^3$ = 28.317 liters |
| | = 28 317 $cm^3$ |
| Force | 1 N = 1 kg·m/$s^2$ = $10^5$ dynes = $10^5$ g·cm/$s^2$ = 0.22481 $lb_f$ |
| | 1 $lb_f$ = 32.174 $lb_m$·ft/$s^2$ = 4.4482 N = 4.4482 x $10^5$ dynes |
| Pressure | 1 atm = 1.01325 x $10^5$ N/$m^2$ (Pa) = 101.325 kPa = 1.01325 bars |
| | = 1.01325 x $10^6$ dynes/$cm^2$ |
| | = 760 mm Hg at 0° C (torr) = 10.333 m $H_2O$ at 4° C |
| | = 14.696 $lb_f$/$in^2$ (psi) = 33.9 ft $H_2O$ at 4°C |
| | 100 kPa = 1 bar |
| Energy | 1 J = 1 N·m = $10^7$ ergs = $10^7$ dyne·cm |
| | = 2.778 x $10^{-7}$ kW·h = 0.23901 cal |
| | = 0.7376 ft·$lb_f$ = 9.47817 x $10^{-4}$ Btu |
| Power | 1 W = 1 J/s = 0.23885 cal/s = 0.7376 ft·$lb_f$/s = 9.47817 x $10^{-4}$ Btu/s = 3.4121 Btu/h |
| | = 1.341 x $10^{-3}$ hp (horsepower) |
| Viscosity | 1 Pa·s = 1 N·s/$m^2$ = 1 kg/m·s |
| | = 10 poise = 10 dynes·s/$cm^2$ = 10 g/cm·s |
| | = $10^3$ cp (centipoise) |
| | = 0.67197 $lb_m$/ft·s = 2419.088 $lb_m$/ft·h |
| Density | 1 kg/$m^3$ = $10^{-3}$ g/$cm^3$ |
| | = 0.06243 $lb_m$/$ft^3$ |
| | $10^3$ kg/$m^3$ = 1 g/$cm^3$ = 62.428 $lb_m$/$ft^3$ |
| Volumetric Flow | 1 $m^3$/s= 35.3145 $ft^3$/s=15,850.2 gal/min (gpm) |
| | 1 gpm = 6.30907 x $10^{-5}$ $m^3$/s=2.22802 x $10^{-3}$ $ft^3$/s=3.7854 liter/min |
| | 1 liter/min=0.26417 gpm |

| | |
|---|---|
| Temperature | $T(^oC) = \frac{5}{9}\left[T(^oF) - 32\right]$ |
| | $T(^oF) = \frac{9}{5}T(^oC) + 32 = 1.8T(^oC) + 32$ |
| Absolute Temperature | T(K) = T(˚C) + 273.15 |
| | T(˚R) = T(˚F) + 459.67 |
| Temperature Interval (ΔT) | 1 C˚ = 1 K = 1.8 F˚ = 1.8 R˚ |
| | 1F˚ = 1 R˚ = (5/9) C˚ = (5/9) K |

## USEFUL QUANTITIES

SG = ρ(20˚C)/$\rho_{water}$ (4˚C)

$\rho_{water}$(4˚C) = 1000 kg/m$^3$ = 62.43 lb$_m$/ft$^3$ = 1.000 g/cm$^3$
$\rho_{water}$(25˚C) = 997.08 kg/m$^3$ = 62.25 lb$_m$/ft$^3$ = 0.99709 g/cm$^3$

g = 9.8066 m/s$^2$ = 980.66 cm/s$^2$ = 32.174 ft/s$^2$

$\mu_{water}$ (25˚C) = 8.937 x 10$^{-4}$ Pa·s = 8.937 x 10$^{-4}$ kg/m·s
= 0.8937 cp =0.8937 x 10$^{-2}$ g/cm·s = 6.005 x 10$^{-4}$ lb$_m$/ft·s

Composition of air:

| | |
|---|---|
| $N_2$ | 78.03% |
| $O_2$ | 20.99% |
| Ar | 0.94% |
| $CO_2$ | 0.03% |
| $H_2$, He, Ne, Kr, Xe | 0.01% |
| | 100.00% |

$M_{air}$ = 29 g/mol = 29 kg/kmol = 29 lb$_m$/lbmole

$\hat{C}_{p,water}$ (25˚C) = 4.182 kJ/kg K = 0.9989 cal/g˚C = 0.9997 Btu/lb$_m$˚F

R = 8.314 m$^3$·Pa/mol·K = 0.08314 liter·bar/mol·K = 0.08206 liter·atm/mol·K
= 62.36 liter·mm Hg/mol·K = 0.7302 ft$^3$·atm/lbmole·˚R
= 10.73 ft$^3$·psia/lbmole·˚R
= 8.314 J/mol·K
= 1.987 cal/mol·K = 1.987 Btu/lbmole·˚R

# Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
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## I. Flow through Smooth Pipes

### A. All Reynolds numbers: Morrison

The correlation from Morrison (2013) fits the smooth pipe data for all Reynolds numbers; beyond $Re = 4000$ this correlation follows the Prandtl equation (see Figure 1; Morrison, equation 7.158). This correlation is explicit in $f$; when flow rate is known, $\Delta p$ may be found directly; when $\Delta p$ is known, $Q$ or $\langle v \rangle$ must be solved for iteratively.

Morrison (2013)
$$f = \left( \frac{0.0076 \left(\frac{3170}{Re}\right)^{0.165}}{1 + \left(\frac{3170}{Re}\right)^{7.0}} \right) + \frac{16}{Re} \qquad (1)$$

### B. $4,000 \le Re \le 1 \times 10^6$: Prandtl

The Prandtl correlation for $f(Re)$ in turbulent flow is not explicit in friction factor and must be solved iteratively except when $f$ is known (Morrison, equation 7.156). This is good only for $Re > 4,000$/

Prandtl
or VonKarman-Nikuradse
(Denn, 1980)
$$\frac{1}{\sqrt{f}} = 4.0 \log\left(Re\sqrt{f}\right) - 0.40 \qquad (2)$$

### C. $4,000 \le Re \le 1 \times 10^6$: A simplified Correlation

For the turbulent regime, an approximate correlation that is much simpler to work with (with a calculator on an exam, for example) is given here and shown in Figure 2 (Morrison, equation 7.157). This is good only for $Re > 4,000$.

Simplified Turbulent
(White, 1974)
$$f = \frac{1.02}{4} (\log Re)^{-2.5} \qquad (3)$$



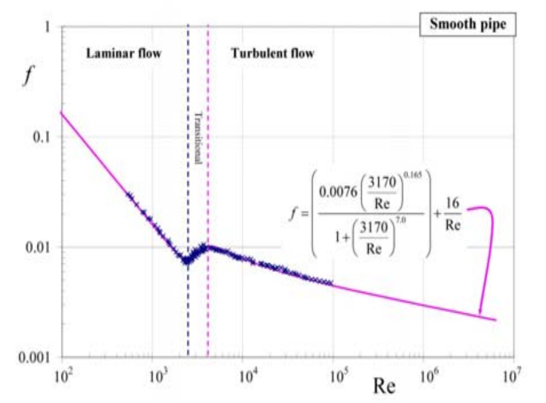


Figure 1: Equation 3 captures smooth pipe friction factor as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are Nikuradse's experimental data for flow in smooth pipes (Nikuradse, 1933). Use beyond $Re = 10^6$ is not recommended; for Re>4000 equation 3 follows the Prandtl equation.

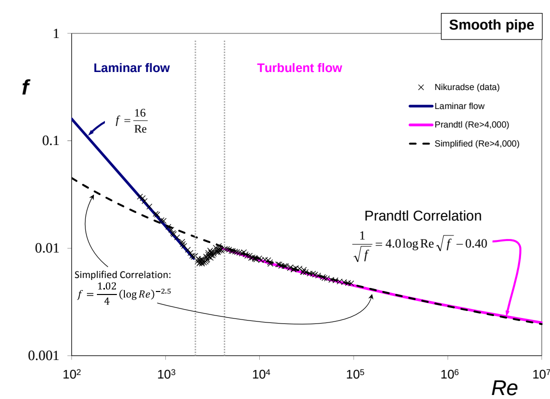


Figure 2: For turbulent flow, the simplified (equation 3) or Prandtl (equation 2) correlations may be used. For work with a calculator, the simplified correlation is perhaps the easiest to work with.

## II. Flow Around a Sphere

### A. All Reynolds Numbers: Morrison

The correlation from Morrison (2013) fits the flow around a sphere for all Reynolds numbers (Figure 3; Morrison equation 8.83); beyond $Re = 10^6$ this correlation follows the curve shown in Figure 3.

Morrison (2013)

$$C_D = \frac{24}{Re} + \frac{2.6\left(\frac{Re}{5.0}\right)}{1+\left(\frac{Re}{5.0}\right)^{1.52}} + \frac{0.411\left(\frac{Re}{263{,}000}\right)^{-7.94}}{1+\left(\frac{Re}{236{,}000}\right)^{-8.00}} + \frac{0.25\left(\frac{Re}{10^6}\right)}{1+\left(\frac{Re}{10^6}\right)} \qquad (4)$$

### Simplified Correlations

The correlations below (Morrison, 2013; equation 8.82) are simpler relationships more suitable to calculator/exam work.

$Re < 2$ $\qquad C_D = \frac{24}{Re}$ $\qquad$ (5)

$0.1 \leq Re \leq 1{,}000$ $\qquad C_D = \frac{24}{Re}(1 + 0.14Re^{0.7})$ $\qquad$ (6)

$1{,}000 \leq Re \leq 2.6 \times 10^5$ $\qquad C_D = 0.445$ $\qquad$ (7)

$2.8 \times 10^5 \leq Re \leq 10^6$ $\qquad \frac{\log C_D}{\left(\frac{Re}{10^6}\right)} = 4.43 \log Re - 27.3$ $\qquad$ (8)



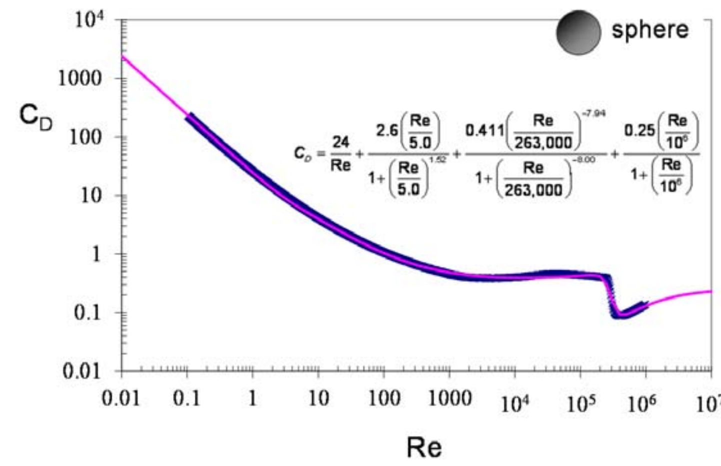


Figure 3: Equation 4 captures flow around a sphere as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are experimental data from White (1974). Use beyond $Re = 10^6$ is not recommended.

**References**

M. Denn, Process Fluid Mechanics (Prentice Hall, Englewood Cliffs, NJ, 1980)
F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press, New York, 2013).
F. M. White, *Viscous Fluid Flow* (McGraw-Hill, Inc.: New York, 1974).

# The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

CM3110 Spring 2002 Faith A. Morrison

Cartesian coordinates

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz} \qquad \text{1-1}$$

Cylindrical coordinates

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z} \qquad \text{1-2}$$

Spherical coordinates

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \tau_{\phi\phi} \end{pmatrix}_{r\theta\phi} =$$

$$\mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} & 2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) \end{pmatrix}_{r\theta\phi} \qquad \text{1-3}$$

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These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *Understanding Rheology* (Oxford University Press: New York, 2001).

From F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge, NY 2013)



Table 1.4. Published friction-loss factors for turbulent flow through valves, fittings, expansions, and contractions

| Fitting | Friction-loss factor, $K_f$ |
|---|---|
| Standard elbow, 45° | 0.35 |
| Standard elbow, 90° | 0.75 |
| Tee used as ell | 1.0 |
| Tee, branch blanked off | 0.4 |
| Return bend | 1.5 |
| Coupling | 0.04 |
| Union | 0.04 |
| Gate valve, wide open | 0.17 |
| Gate valve, half open | 4.5 |
| Globe valve, bevel seat, wide open | 6.0 |
| Globe valve, bevel seat, half open | 9.5 |
| Check valve, ball | 70.0 |
| Check valve, swing | 2.0 |
| Water meter, disk | 7.0 |
| Expansion from $A_1$ to $A_2$ | $\left(1-\frac{A_1}{A_2}\right)^2$ |
| Contraction from $A_1$ to $A_2$ | $0.55\left(1-\frac{A_2}{A_1}\right)$ |

*Source: Perry's Handbook* [132]

Table 1.5. Friction-loss factors $K_f$ for laminar flow through selected valves, fittings, expansions and contractions

| | $K_f$ | | | | | |
|---|---|---|---|---|---|---|
| Fitting | $\mathrm{Re}_l = 50$ | 100 | 200 | 400 | 1,000 | Turbulent |
| Elbow, 90° | 17 | 7 | 2.5 | 1.2 | 0.85 | 0.75 |
| Tee | 9 | 4.8 | 3.0 | 2.0 | 1.4 | 1.0 |
| Globe valve | 28 | 22 | 17 | 14 | 10 | 6.0 |
| Check valve, swing | 55 | 17 | 9 | 5.8 | 3.2 | 2.0 |
| Expansion from $A_1$ to $A_2$ | $2\left(1-\frac{A_1}{A_2}\right)^2$ | | | | | $\left(1-\frac{A_1}{A_2}\right)^2$ |
| Contraction from $A_1$ to $A_2$ | $\frac{0.55}{0.5}\left(1-\frac{A_2}{A_1}\right)$ | | | | | $0.55\left(1-\frac{A_2}{A_1}\right)$ |

*Source: Perry's Handbook* [132]

# Energy Balance Notes

*CM2110/CM3110*
*Professor Faith A. Morrison*
*December 4, 2008*

**References**

1. (FR) R. M. Felder, and R. W. Rousseau, *Elementary Principles of Chemical Processe,* 2nd Edition (Wiley, NY: 1986).
2. (G) C. J. Geankoplis, *Transport Processes and Unit Operations*, 3rd Edition (Prentice Hall: Englewood Cliffs, NJ, 1993).

1. Closed System (note: $\Delta = \Sigma_{final} - \Sigma_{initial}$ )
   - $\Delta E_k + \Delta E_p + \Delta U = Q_{in} + W_{on}$ (FR)
   - Is it adiabatic? (if yes, $Q_{in}$=0)
   - Are there moving parts, e.g. do the walls move? (if no, $W_{on}$=0)
   - Is the *system* moving? (if no, $\Delta E_k$=0)
   - Is there a change in elevation of the *system*? (if no, $\Delta E_p = 0$ )
   - Does T, phase, or chemical composition change? (if no to all, $\Delta U$=0)

2. Open System (the fluid is the system) (note: $\Delta = \Sigma_{out} - \Sigma_{in}$ )
   1. Is it a Mechanical Energy Balance (MEB) problem?
      (turbulent, α=1; laminar, α=0.5; *F*= total frictional loss)
      - $\frac{\Delta P}{\rho} + \frac{1}{2\alpha}\Delta v^2 + g\,\Delta z + F = \frac{W_{s,on\,fluid}}{m}$ (FR)
      - $\frac{\Delta P}{\rho} + \frac{1}{2\alpha}\Delta v^2 + g\,\Delta z + F = -W_{s,by\,fluid}$ (G)

      The mechanical energy balance is only valid for systems for which the following is true:

      i. single-input, single output
      ii. small or zero $Q_{in}$
      iii. incompressible fluid (ρ = constant)
      iv. small or zero $\Delta T$

   2. Is it a regular open system balance?
      - $\Delta E_k + \Delta E_p + \Delta H = Q_{in} + W_{s,on}$ (FR)
      - Is it adiabatic? (if yes, $Q_{in}$=0)
      - Are there moving parts, e.g. pump, turbine, mixing shaft? (if no, $W_{s,on}$=0)





- Does the average velocity of the fluid change between the input and the output? (if no, $\Delta E_k$=0); remember $<v> = v_{av} = v = \frac{v(\frac{m^3}{s})}{A(m^2)}$, where $(\frac{m^3}{s})$ is volumetric flow rate.
- Is there a change in elevation of the system between the input and the output? (if no, $\Delta E_p = 0$ )
- Does T, phase, chemical composition, or **P change**? (if no to all, $\Delta H$=0)





# Calculating Internal Energy

1. Constant T, **P changes only**
   (a) real gases => look it up in a table (e.g. **steam**, Tables B4, B5, B6)
   (b) ideal gases => $\Delta\hat{u}$=0
   (c) liquids, solids => $\Delta\hat{u}$=0
2. Constant P, **T changes only**
   (a) real gases => look it up in a table (e.g. **steam**),
   or, if *V* is constant, $\Delta\hat{u} = \int_{T_1}^{T_2} \hat{C}v\,(T)\,dT$
   (b) ideal gases => $\Delta\hat{u} = \int_{T_1}^{T_2} \hat{C}v\,(T)\,dT$
   also, $\hat{C}_p = \hat{C}_v + R$
   (c) liquids, solids $\Delta\hat{u} = \int_{T_1}^{T_2} \hat{C}v\,(T)\,dT$
   also $\hat{C}_p \approx \hat{C}_v$
3. Constant T, P, **phase changes**
   (a) real gases => look it up in a table (e.g. **steam**)
   (b) liquid to vapor => $\Delta\hat{u} = \Delta\hat{H}_{vap}(T) - P\Delta\hat{V}_{vap} \approx \Delta\hat{H}_{vap} - RT$
   (c) solid to vapor => $\Delta\hat{u} = \Delta\hat{H}_{sub}(T) - P\Delta\hat{V}_{sub} \approx \Delta\hat{H}_{sub} - RT$
   (d) solid to liquid => $\Delta\hat{u} = \Delta\hat{H}_{melt}(T) - P\Delta\hat{V}_{melt} \approx \Delta\hat{H}_{melt}$
4. Constant T, P, **mixing occurs**
   (a) gases => $\Delta\hat{u}$ =0
   (b) similar liquids => $\Delta\hat{u}$ =0
   (c) dissimilar liquids/solids => $\Delta\hat{u} = \Delta\hat{H}_{solution}$, Table 8.5-1, FR page 380
   Note: be careful with units, $\Delta\hat{H}_{solution}$ [=] $\frac{kJ}{mole\ solute}$
5. Constant T, P, **reaction occurs:** $\Delta\hat{u} = \Delta\hat{H}_{rxn}$





# Calculating Enthalpy

1. Constant T, **P changes only** (Note: Since T is constant, $\hat{U}$ does not change.)
   (a) real gases - look it up in a table (e.g. **steam**, Tables B4, B5, B6)
   (b) ideal gases

   $$\hat{H} = \hat{U} + P\hat{V} \quad (1)$$
   $$= \hat{U} + RT \quad (2)$$
   $$(\hat{H}_2 - \hat{H}_1) = (\hat{U}_2 - \hat{U}_1) + R(T_2 - T_1) \quad (3)$$
   $$\Delta\hat{H} = \Delta\hat{U} = 0 \quad (4)$$

   (c) liquids, solids

   $$\hat{H} = \hat{U} + P\hat{V} \quad (5)$$
   $$\Delta\hat{H} = \Delta(P\hat{V}) \quad (6)$$
   $$\hat{V} \approx \text{constant wrt } P \quad (7)$$
   $$\Delta\hat{H} = \hat{V}(\Delta P) \quad (8)$$

2. Constant P, **T changes only**
   (a) real gases => look it up in a table to be most accurate (e.g. **steam**),
   otherwise $\Delta\hat{H} = \int_{T_1}^{T_2} \hat{C}_p\,(T)\,dT$
   (b) ideal gases => $\Delta\hat{H} = \int_{T_1}^{T_2} \hat{C}_p\,(T)\,dT$
   (c) liquids, solids => $\Delta\hat{H} = \int_{T_1}^{T_2} \hat{C}_p\,(T)\,dT$
3. Constant T, P, **phase changes**
   (a) liquid to vapor => $\Delta\hat{H} = \Delta\hat{H}_{vap}(T)$
   Note: $\frac{d\,\ln P*}{d \ln 1/T} = \frac{\Delta\hat{H}vap}{R}$ Clapeyron equation
   (b) solid to vapor => $\Delta\hat{H} = \Delta\hat{H}_{sub}(T)$
   (c) solid to liquid => $\Delta\hat{H} = \Delta\hat{H}_{melt}(T)$
4. Constant T, P, **mixing occurs**
   (a) gases => $\Delta\hat{H}$ =0
   (b) similar liquids => $\Delta\hat{H}$ =0
   (c) dissimilar liquids/solids => $\Delta\hat{H} = \Delta\hat{H}_{solution}$, Table 8.5-1, FR page 380
   Note: be careful with units, $\Delta\hat{H}_{solution}$ [=] $\frac{J}{mole\ solute}$
5. Constant T, P, **reaction occurs::** $\Delta\hat{u} = \Delta\hat{H}_{rxn}$

# Problem-Solving Strategies for Energy Balances

1. Write down *neatly* everything you are doing so that you and the grader both understand better what you are thinking.

2. Draw your flow sheet Large. Leave yourself plenty of room to add information to the drawing. Draw it over if it becomes too crowded. Do not erase; start a new sheet. You may want to come back to the original information.

3. Always write complete units with quantities, e.g., for mole fraction A the units are $\frac{moles\ A}{moles\ total}$; quantities are meaningless without the units.

4. What are you looking for? What balance can help you find it?

    a. Is a mass balance necessary?
    b. Is it an open or a closed system?
    c. Is it a mechanical energy balance problem?

5. Convert inconvenient units, e.g., convert volumes and volume fractions into moles or masses, since mass is conserved and volume is not. Also, convert dew points and other similar information (e.g. percent humidity, molal saturation, etc.) to compositions if possible.

6. Do you have a piece of information you do not know what to do with? What is its definition? Look it up in the index, if necessary. Write it with its proper units and try to interpret how it impacts the problem.

7. What has remained constant in the problem? Is it isothermal (T constant)? Isobaric (P constant)? Constant $V$ or $\hat{V}$? Adiabatic ($Q$=0)? Is the mass flow constant? Is the volumetric flow constant? Is the heat flow constant or known?

8. Remember that if a system is ***saturated***, you know a great deal about it:

    a. If it is a *pure component*, you only need to know the phase (i.e. solid, liquid, vapor) and *one* of the following to know everything about the stream: $T$, $P$, $\hat{V}$, $\hat{U}$, $\hat{H}$.
    b. If it is a *mixture*, Raoult's law applies to each component, $y_iP = x_iP_i^*(T)$.

9. When looking for $\hat{U}$, $\hat{H}$, or $\hat{V}$, always take it from a table, if it is available. It *is* available in a table for water/steam:

    a. Table B.4, FR page 628, "Properties of *Saturated* Steam," sorted by Temperature
    b. Table B.5, FR page 630, "Properties of *Saturated* Steam,'" sorted by Pressure





    c. Table B.6, FR page 636, "Properties of *Superheated* Steam," presented in a grid of pressure and temperature. Saturated steam properties are also presented in the first two columns of Table B.6, but the steps in pressure are large, and therefore Tables B.4 and B.5 are more accurate for saturated steam properties at lower T and P.

10. If the problem is complex, break it down into smaller pieces and draw separate flow-sheets which correspond to the smaller pieces.

11. Name unknown streams, compositions, and enthalpies or internal energies. See if there are few unknowns which can be solved for. Try different methods of naming the unknowns if the first way you think of does not turn out to be convenient.

12. Check for forgotten relations:

    a. mass balance
    b. mole fractions and mass fractions sum to 1.
    c. If a stream is just split, with no special process unit present, the mole or mass fractions are the same in all streams before and after the split.
    d. For a fixed, closed system, $V$, $\hat{V}$, and mass are constant.

13. The last step is to answer the question. Always present your answer with the correct number of significant figures and a box around it.

The **Equation of Energy** in Cartesian, cylindrical, and spherical coordinates for Newtonian fluids of constant density, with source term $S$. Source could be electrical energy due to current flow, chemical energy, etc. Two cases are presented: the general case where thermal conductivity may be a function of temperature (vector flux $\underline{\tilde{q}} = \underline{q}/A$ appears in the equations); and the more usual case, where thermal conductivity is constant.

Fall 2013 Faith A. Morrison, Michigan Technological University

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**Microscopic energy balance**, in terms of flux; Gibbs notation

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = -\nabla \cdot \underline{\tilde{q}} + S$$

**Microscopic energy balance**, in terms of flux; Cartesian coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = -\left( \frac{\partial \tilde{q}_x}{\partial x} + \frac{\partial \tilde{q}_y}{\partial y} + \frac{\partial \tilde{q}_z}{\partial z} \right) + S$$

**Microscopic energy balance**, in terms of flux; cylindrical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = -\left( \frac{1}{r} \frac{\partial (r \tilde{q}_r)}{\partial r} + \frac{1}{r} \frac{\partial \tilde{q}_\theta}{\partial \theta} + \frac{\partial \tilde{q}_z}{\partial z} \right) + S$$

**Microscopic energy balance**, in terms of flux; spherical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = -\left( \frac{1}{r^2} \frac{\partial (r^2 \tilde{q}_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tilde{q}_\theta sin\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{q}_\phi}{\partial \phi} \right) + S$$

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**Fourier's law of heat conduction**, Gibbs notation: $\underline{\tilde{q}} = -k \nabla T$

**Fourier's law of heat conduction**, Cartesian coordinates: $\begin{pmatrix} \tilde{q}_x \\ \tilde{q}_y \\ \tilde{q}_z \end{pmatrix}_{xyz} = \begin{pmatrix} -k \frac{\partial T}{\partial x} \\ -k \frac{\partial T}{\partial y} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{xyz}$

**Fourier's law of heat conduction**, cylindrical coordinates: $\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_z \end{pmatrix}_{xyz} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{r\theta z}$

**Fourier's law of heat conduction**, spherical coordinates: $\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_\phi \end{pmatrix}_{xyz} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -\frac{k}{r \sin \theta} \frac{\partial T}{\partial \phi} \end{pmatrix}_{r\theta\phi}$

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## The **Equation of Energy** for systems with **constant $k$**

**Microscopic energy balance**, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

**Microscopic energy balance**, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S$$

**Microscopic energy balance**, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S$$

**Microscopic energy balance**, constant thermal conductivity; spherical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right)$$

$$= k \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S$$

Reference: F. A. Morrison, “Web Appendix to *An Introduction to Fluid Mechanics*,” Cambridge University Press, New York, 2013. On the web at www.chem.mtu.edu/~fmorriso/IFM_WebAppendixCD2013.pdf

## A.2-9 Properties of Saturated Steam and Water (Steam Table), SI Units

| Temperature (°C) | Vapor Pressure (kPa) | Specific Volume ($m^3/kg$) | | Enthalpy (kJ/kg) | | Entropy (kJ/kg · K) | |
|---|---|---|---|---|---|---|---|
| | | Liquid | Sat'd Vapor | Liquid | Sat'd Vapor | Liquid | Sat'd Vapor |
| 0.01 | 0.6113 | 0.0010002 | 206.136 | 0.00 | 2501.4 | 0.0000 | 9.1562 |
| 3 | 0.7577 | 0.0010001 | 168.132 | 12.57 | 2506.9 | 0.0457 | 9.0773 |
| 6 | 0.9349 | 0.0010001 | 137.734 | 25.20 | 2512.4 | 0.0912 | 9.0003 |
| 9 | 1.1477 | 0.0010003 | 113.386 | 37.80 | 2517.9 | 0.1362 | 8.9253 |
| 12 | 1.4022 | 0.0010005 | 93.784 | 50.41 | 2523.4 | 0.1806 | 8.8524 |
| 15 | 1.7051 | 0.0010009 | 77.926 | 62.99 | 2528.9 | 0.2245 | 8.7814 |
| 18 | 2.0640 | 0.0010014 | 65.038 | 75.58 | 2534.4 | 0.2679 | 8.7123 |
| 21 | 2.487 | 0.0010020 | 54.514 | 88.14 | 2539.9 | 0.3109 | 8.6450 |
| 24 | 2.985 | 0.0010027 | 45.883 | 100.70 | 2545.4 | 0.3534 | 8.5794 |
| 25 | 3.169 | 0.0010029 | 43.360 | 104.89 | 2547.2 | 0.3674 | 8.5580 |
| 27 | 3.567 | 0.0010035 | 38.774 | 113.25 | 2550.8 | 0.3954 | 8.5156 |
| 30 | 4.246 | 0.0010043 | 32.894 | 125.79 | 2556.3 | 0.4369 | 8.4533 |
| 33 | 5.034 | 0.0010053 | 28.011 | 138.33 | 2561.7 | 0.4781 | 8.3927 |
| 36 | 5.947 | 0.0010063 | 23.940 | 150.86 | 2567.1 | 0.5188 | 8.3336 |
| 40 | 7.384 | 0.0010078 | 19.523 | 167.57 | 2574.3 | 0.5725 | 8.2570 |
| 45 | 9.593 | 0.0010099 | 15.258 | 188.45 | 2583.2 | 0.6387 | 8.1648 |
| 50 | 12.349 | 0.0010121 | 12.032 | 209.33 | 2592.1 | 0.7038 | 8.0763 |
| 55 | 15.758 | 0.0010146 | 9.568 | 230.23 | 2600.9 | 0.7679 | 7.9913 |
| 60 | 19.940 | 0.0010172 | 7.671 | 251.13 | 2609.6 | 0.8312 | 7.9096 |
| 65 | 25.03 | 0.0010199 | 6.197 | 272.06 | 2618.3 | 0.8935 | 7.8310 |
| 70 | 31.19 | 0.0010228 | 5.042 | 292.98 | 2626.8 | 0.9549 | 7.7553 |
| 75 | 38.58 | 0.0010259 | 4.131 | 313.93 | 2635.3 | 1.0155 | 7.6824 |
| 80 | 47.39 | 0.0010291 | 3.407 | 334.91 | 2643.7 | 1.0753 | 7.6122 |
| 85 | 57.83 | 0.0010325 | 2.828 | 355.90 | 2651.9 | 1.1343 | 7.5445 |
| 90 | 70.14 | 0.0010360 | 2.361 | 376.92 | 2660.1 | 1.1925 | 7.4791 |
| 95 | 84.55 | 0.0010397 | 1.9819 | 397.96 | 2668.1 | 1.2500 | 7.4159 |
| 100 | 101.35 | 0.0010435 | 1.6729 | 419.04 | 2676.1 | 1.3069 | 7.3549 |
| 105 | 120.82 | 0.0010475 | 1.4194 | 440.15 | 2683.8 | 1.3630 | 7.2958 |
| 110 | 143.27 | 0.0010516 | 1.2102 | 461.30 | 2691.5 | 1.4185 | 7.2387 |
| 115 | 169.06 | 0.0010559 | 1.0366 | 482.48 | 2699.0 | 1.4734 | 7.1833 |
| 120 | 198.53 | 0.0010603 | 0.8919 | 503.71 | 2706.3 | 1.5276 | 7.1296 |
| 125 | 232.1 | 0.0010649 | 0.7706 | 524.99 | 2713.5 | 1.5813 | 7.0775 |
| 130 | 270.1 | 0.0010697 | 0.6685 | 546.31 | 2720.5 | 1.6344 | 7.0269 |
| 135 | 313.0 | 0.0010746 | 0.5822 | 567.69 | 2727.3 | 1.6870 | 6.9777 |
| 140 | 316.3 | 0.0010797 | 0.5089 | 589.13 | 2733.9 | 1.7391 | 6.9299 |
| 145 | 415.4 | 0.0010850 | 0.4463 | 610.63 | 2740.3 | 1.7907 | 6.8833 |
| 150 | 475.8 | 0.0010905 | 0.3928 | 632.20 | 2746.5 | 1.8418 | 6.8379 |
| 155 | 543.1 | 0.0010961 | 0.3468 | 653.84 | 2752.4 | 1.8925 | 6.7935 |
| 160 | 617.8 | 0.0011020 | 0.3071 | 675.55 | 2758.1 | 1.9427 | 6.7502 |
| 165 | 700.5 | 0.0011080 | 0.2727 | 697.34 | 2763.5 | 1.9925 | 6.7078 |
| 170 | 791.7 | 0.0011143 | 0.2428 | 719.21 | 2768.7 | 2.0419 | 6.6663 |
| 175 | 892.0 | 0.0011207 | 0.2168 | 741.17 | 2773.6 | 2.0909 | 6.6256 |
| 180 | 1002.1 | 0.0011274 | 0.19405 | 763.22 | 2778.2 | 2.1396 | 6.5857 |
| 190 | 1254.4 | 0.0011414 | 0.15654 | 807.62 | 2786.4 | 2.2359 | 6.5079 |
| 200 | 1553.8 | 0.0011565 | 0.12736 | 852.45 | 2793.2 | 2.3309 | 6.4323 |
| 225 | 2548 | 0.0011992 | 0.07849 | 966.78 | 2803.3 | 2.5639 | 6.2503 |
| 250 | 3973 | 0.0012512 | 0.05013 | 1085.36 | 2801.5 | 2.7927 | 6.0730 |
| 275 | 5942 | 0.0013168 | 0.03279 | 1210.07 | 2785.0 | 3.0208 | 5.8938 |
| 300 | 8581 | 0.0010436 | 0.02167 | 1344.0 | 2749.0 | 3.2534 | 5.7045 |

*Source:* Abridged from J. H. Keenan, F. G. Keyes, P. G. Hill, and J. G. Moore, *Steam Tables—Metric Units*. New York: John Wiley & Sons, Inc., 1969. Reprinted by permission of John Wiley & Sons, Inc.

## A.2-10 Properties of Superheated Steam (Steam Table), SI Units ($v$, specific volume, $m^3/kg$; $H$, enthalpy, kJ/kg; $s$, entropy, kJ/kg · K)

| Absolute Pressure, kPa (Sat. Temp., °C) | | Temperature (°C) 100 | 150 | 200 | 250 | 300 | 360 | 420 | 500 |
|---|---|---|---|---|---|---|---|---|---|
| | $v$ | 17.196 | 19.512 | 21.825 | 24.136 | 26.445 | 29.216 | 31.986 | 35.679 |
| 10 | $H$ | 2687.5 | 2783.0 | 2879.5 | 2977.3 | 3076.5 | 3197.6 | 3320.9 | 3489.1 |
| (45.81) | $s$ | 8.4479 | 8.6882 | 8.9038 | 9.1002 | 9.2813 | 9.4821 | 9.6682 | 9.8978 |
| | $v$ | 3.418 | 3.889 | 4.356 | 4.820 | 5.284 | 5.839 | 6.394 | 7.134 |
| 50 | $H$ | 2682.5 | 2780.1 | 2877.7 | 2976.0 | 3075.5 | 3196.8 | 3320.4 | 3488.7 |
| (81.33) | $s$ | 7.6947 | 7.9401 | 8.1580 | 8.3556 | 8.5373 | 8.7385 | 8.9249 | 9.1546 |
| | $v$ | 2.270 | 2.587 | 2.900 | 3.211 | 3.520 | 3.891 | 4.262 | 4.755 |
| 75 | $H$ | 2679.4 | 2778.2 | 2876.5 | 2975.2 | 3074.9 | 3196.4 | 3320.0 | 3488.4 |
| (91.78) | $s$ | 7.5009 | 7.7496 | 7.9690 | 8.1673 | 8.3493 | 8.5508 | 8.7374 | 8.9672 |
| | $v$ | 1.6958 | 1.9364 | 2.172 | 2.406 | 2.639 | 2.917 | 3.195 | 3.565 |
| 100 | $H$ | 2672.2 | 2776.4 | 2875.3 | 2974.3 | 3074.3 | 3195.9 | 3319.6 | 3488.1 |
| (99.63) | $s$ | 7.3614 | 7.6134 | 7.8343 | 8.0333 | 8.2158 | 8.4175 | 8.6042 | 8.8342 |
| | $v$ | | 1.2853 | 1.4443 | 1.6012 | 1.7570 | 1.9432 | 2.129 | 2.376 |
| 150 | $H$ | | 2772.6 | 2872.9 | 2972.7 | 3073.1 | 3195.0 | 3318.9 | 3487.6 |
| (111.37) | $s$ | | 7.4193 | 7.6433 | 7.8438 | 8.0720 | 8.2293 | 8.4163 | 8.6466 |
| | $v$ | | 0.4708 | 0.5342 | 0.5951 | 0.6548 | 0.7257 | 0.7960 | 0.8893 |
| 400 | $H$ | | 2752.8 | 2860.5 | 2964.2 | 3066.8 | 3190.3 | 3315.3 | 3484.9 |
| (143.63) | $s$ | | 6.9299 | 7.1706 | 7.3789 | 7.5662 | 7.7712 | 7.9598 | 8.1913 |
| | $v$ | | | 0.2999 | 0.3363 | 0.3714 | 0.4126 | 0.4533 | 0.5070 |
| 700 | $H$ | | | 2844.8 | 2953.6 | 3059.1 | 3184.7 | 3310.9 | 3481.7 |
| (164.97) | $s$ | | | 6.8865 | 7.1053 | 7.2979 | 7.5063 | 7.6968 | 7.9299 |
| | $v$ | | | 0.2060 | 0.2327 | 0.2579 | 0.2873 | 0.3162 | 0.3541 |
| 1000 | $H$ | | | 2827.9 | 2942.6 | 3051.2 | 3178.9 | 3306.5 | 3478.5 |
| (179.91) | $s$ | | | 6.6940 | 6.9247 | 7.1229 | 7.3349 | 7.5275 | 7.7622 |
| | $v$ | | | 0.13248 | 0.15195 | 0.16966 | 0.18988 | 0.2095 | 0.2352 |
| 1500 | $H$ | | | 2796.8 | 2923.3 | 3037.6 | 3.1692 | 3299.1 | 3473.1 |
| (198.32) | $s$ | | | 6.4546 | 6.7090 | 6.9179 | 7.1363 | 7.3323 | 7.5698 |
| | $v$ | | | | 0.11144 | 0.12547 | 0.14113 | 0.15616 | 0.17568 |
| 2000 | $H$ | | | | 2902.5 | 3023.5 | 3159.3 | 3291.6 | 3467.6 |
| (212.42) | $s$ | | | | 6.5453 | 6.7664 | 6.9917 | 7.1915 | 7.4317 |
| | $v$ | | | | 0.08700 | 0.09890 | 0.11186 | 0.12414 | 0.13998 |
| 2500 | $H$ | | | | 2880.1 | 3008.8 | 3149.1 | 3284.0 | 3462.1 |
| (223.99) | $s$ | | | | 6.4085 | 6.6438 | 6.8767 | 7.0803 | 7.3234 |
| | $v$ | | | | 0.07058 | 0.08114 | 0.09233 | 0.10279 | 0.11619 |
| 3000 | $H$ | | | | 2855.8 | 2993.5 | 3138.7 | 3276.3 | 3456.5 |
| (233.90) | $s$ | | | | 6.2872 | 6.5390 | 6.7801 | 6.9878 | 7.2338 |

*Source:* Abridged from J. H. Keenan, F. G. Keyes, P. G. Hill, and J. G. Moore, *Steam Tables—Metric Units.* New York: John Wiley & Sons, Inc., 1969. Reprinted by permission of John Wiley & Sons, Inc.

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## A.2-11 Heat-Transfer Properties of Liquid Water, SI Units

| T (°C) | T (K) | $\rho$ ($kg/m^3$) | $c_p$ ($kJ/kg \cdot K$) | $\mu \times 10^3$ ($Pa \cdot s$, or $kg/m \cdot s$) | k ($W/m \cdot K$) | $N_{Pr}$ | $\beta \times 10^4$ (1/K) | $(g\beta\rho^2/\mu^2) \times 10^{-8}$ ($1/K \cdot m^3$) |
|---|---|---|---|---|---|---|---|---|
| 0 | 273.2 | 999.6 | 4.229 | 1.786 | 0.5694 | 13.3 | −0.630 | |
| 15.6 | 288.8 | 998.0 | 4.187 | 1.131 | 0.5884 | 8.07 | 1.44 | 10.93 |
| 26.7 | 299.9 | 996.4 | 4.183 | 0.860 | 0.6109 | 5.89 | 2.34 | 30.70 |
| 37.8 | 311.0 | 994.7 | 4.183 | 0.682 | 0.6283 | 4.51 | 3.24 | 68.0 |
| 65.6 | 338.8 | 981.9 | 4.187 | 0.432 | 0.6629 | 2.72 | 5.04 | 256.2 |
| 93.3 | 366.5 | 962.7 | 4.229 | 0.3066 | 0.6802 | 1.91 | 6.66 | 642 |
| 121.1 | 394.3 | 943.5 | 4.271 | 0.2381 | 0.6836 | 1.49 | 8.46 | 1300 |
| 148.9 | 422.1 | 917.9 | 4.312 | 0.1935 | 0.6836 | 1.22 | 10.08 | 2231 |
| 204.4 | 477.6 | 858.6 | 4.522 | 0.1384 | 0.6611 | 0.950 | 14.04 | 5308 |
| 260.0 | 533.2 | 784.9 | 4.982 | 0.1042 | 0.6040 | 0.859 | 19.8 | 11 030 |
| 315.6 | 588.8 | 679.2 | 6.322 | 0.0862 | 0.5071 | 1.07 | 31.5 | 19 260 |

## A.2-11 Heat-Transfer Properties of Liquid Water, English Units

| T (°F) | $\rho$ $\left(\frac{lb_m}{ft^3}\right)$ | $c_p$ $\left(\frac{btu}{lb_m \cdot °F}\right)$ | $\mu \times 10^3$ $\left(\frac{lb_m}{ft \cdot s}\right)$ | k $\left(\frac{btu}{h \cdot ft \cdot °F}\right)$ | $N_{Pr}$ | $\beta \times 10^4$ (1/°R) | $(g\beta\rho^2/\mu^2) \times 10^{-6}$ ($1/°R \cdot ft^3$) |
|---|---|---|---|---|---|---|---|
| 32 | 62.4 | 1.01 | 1.20 | 0.329 | 13.3 | −0.350 | |
| 60 | 62.3 | 1.00 | 0.760 | 0.340 | 8.07 | 0.800 | 17.2 |
| 80 | 62.2 | 0.999 | 0.578 | 0.353 | 5.89 | 1.30 | 48.3 |
| 100 | 62.1 | 0.999 | 0.458 | 0.363 | 4.51 | 1.80 | 107 |
| 150 | 61.3 | 1.00 | 0.290 | 0.383 | 2.72 | 2.80 | 403 |
| 200 | 60.1 | 1.01 | 0.206 | 0.393 | 1.91 | 3.70 | 1010 |
| 250 | 58.9 | 1.02 | 0.160 | 0.395 | 1.49 | 4.70 | 2045 |
| 300 | 57.3 | 1.03 | 0.130 | 0.395 | 1.22 | 5.60 | 3510 |
| 400 | 53.6 | 1.08 | 0.0930 | 0.382 | 0.950 | 7.80 | 8350 |
| 500 | 49.0 | 1.19 | 0.0700 | 0.349 | 0.859 | 11.0 | 17 350 |
| 600 | 42.4 | 1.51 | 0.0579 | 0.293 | 1.07 | 17.5 | 30 300 |

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## A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), SI Units

| $T$ (°C) | $T$ (K) | $\rho$ (kg/m³) | $c_p$ (kJ/kg·K) | $\mu \times 10^5$ (Pa·s, or kg/m·s) | $k$ (W/m·K) | $N_{Pr}$ | $\beta \times 10^3$ (1/K) | $g\beta\rho^2/\mu^2$ (1/K·m³) |
|---|---|---|---|---|---|---|---|---|
| −17.8 | 255.4 | 1.379 | 1.0048 | 1.62 | 0.02250 | 0.720 | 3.92 | $2.79 \times 10^8$ |
| 0 | 273.2 | 1.293 | 1.0048 | 1.72 | 0.02423 | 0.715 | 3.65 | $2.04 \times 10^8$ |
| 10.0 | 283.2 | 1.246 | 1.0048 | 1.78 | 0.02492 | 0.713 | 3.53 | $1.72 \times 10^8$ |
| 37.8 | 311.0 | 1.137 | 1.0048 | 1.90 | 0.02700 | 0.705 | 3.22 | $1.12 \times 10^8$ |
| 65.6 | 338.8 | 1.043 | 1.0090 | 2.03 | 0.02925 | 0.702 | 2.95 | $0.775 \times 10^8$ |
| 93.3 | 366.5 | 0.964 | 1.0090 | 2.15 | 0.03115 | 0.694 | 2.74 | $0.534 \times 10^8$ |
| 121.1 | 394.3 | 0.895 | 1.0132 | 2.27 | 0.03323 | 0.692 | 2.54 | $0.386 \times 10^8$ |
| 148.9 | 422.1 | 0.838 | 1.0174 | 2.37 | 0.03531 | 0.689 | 2.38 | $0.289 \times 10^8$ |
| 176.7 | 449.9 | 0.785 | 1.0216 | 2.50 | 0.03721 | 0.687 | 2.21 | $0.214 \times 10^8$ |
| 204.4 | 477.6 | 0.740 | 1.0258 | 2.60 | 0.03894 | 0.686 | 2.09 | $0.168 \times 10^8$ |
| 232.2 | 505.4 | 0.700 | 1.0300 | 2.71 | 0.04084 | 0.684 | 1.98 | $0.130 \times 10^8$ |
| 260.0 | 533.2 | 0.662 | 1.0341 | 2.80 | 0.04258 | 0.680 | 1.87 | $0.104 \times 10^8$ |

## A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), English Units

| $T$ (°F) | $\rho$ $\left(\frac{lb_m}{ft^3}\right)$ | $c_p$ $\left(\frac{btu}{lb_m \cdot °F}\right)$ | $\mu$ (centipoise) | $k$ $\left(\frac{btu}{h \cdot ft \cdot °F}\right)$ | $N_{Pr}$ | $\beta \times 10^3$ (1/°R) | $g\beta\rho^2/\mu^2$ (1/°R·ft³) |
|---|---|---|---|---|---|---|---|
| 0 | 0.0861 | 0.240 | 0.0162 | 0.0130 | 0.720 | 2.18 | $4.39 \times 10^6$ |
| 32 | 0.0807 | 0.240 | 0.0172 | 0.0140 | 0.715 | 2.03 | $3.21 \times 10^6$ |
| 50 | 0.0778 | 0.240 | 0.0178 | 0.0144 | 0.713 | 1.96 | $2.70 \times 10^6$ |
| 100 | 0.0710 | 0.240 | 0.0190 | 0.0156 | 0.705 | 1.79 | $1.76 \times 10^6$ |
| 150 | 0.0651 | 0.241 | 0.0203 | 0.0169 | 0.702 | 1.64 | $1.22 \times 10^6$ |
| 200 | 0.0602 | 0.241 | 0.0215 | 0.0180 | 0.694 | 1.52 | $0.840 \times 10^6$ |
| 250 | 0.0559 | 0.242 | 0.0227 | 0.0192 | 0.692 | 1.41 | $0.607 \times 10^6$ |
| 300 | 0.0523 | 0.243 | 0.0237 | 0.0204 | 0.689 | 1.32 | $0.454 \times 10^6$ |
| 350 | 0.0490 | 0.244 | 0.0250 | 0.0215 | 0.687 | 1.23 | $0.336 \times 10^6$ |
| 400 | 0.0462 | 0.245 | 0.0260 | 0.0225 | 0.686 | 1.16 | $0.264 \times 10^6$ |
| 450 | 0.0437 | 0.246 | 0.0271 | 0.0236 | 0.674 | 1.10 | $0.204 \times 10^6$ |
| 500 | 0.0413 | 0.247 | 0.0280 | 0.0246 | 0.680 | 1.04 | $0.163 \times 10^6$ |

*Source:* National Bureau of Standards, *Circular* 461C, 1947; 564, 1955; NBS–NACA, *Tables of Thermal Properties of Gases*, 1949; F. G. Keyes, *Trans. A.S.M.E.*, **73**, 590, 597 (1951); **74**, 1303 (1952); D. D. Wagman, *Selected Values of Chemical Thermodynamic Properties*. Washington, D.C.: National Bureau of Standards, 1953.

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