The Mechanical Energy Balance is a macroscopic analysis.

- **It is limited:**
  1. single-input, single output
  2. Steady state
  3. Constant density (incompressible fluid)
  4. Temperature approximately constant
  5. No phase change, no chemical rxn
  6. Insignificant amounts of heat transferred

- **It cannot determine flow patterns**
- **It does not model momentum exchanges**
- **It cannot be adapted to systems other than those for which it was designed**

Energy balances (the MEB) can only take us so far with fluids modeling (due to assumptions).

To understand complex flows, we must use the **MOMENTUM** balance.
Momentum Balance: Newton’s 2nd Law of Motion

\[ f = ma \]

Phys2100: apply to individual bodies
CM3110: apply to a continuum

See also: http://youtu.be/6KKNjIPgGto

Fluid Mechanics

\[ \tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \]

- Continuum (density, velocity, stress fields)
- Control volume
- Stress in a fluid at a point (stress tensor)
- Stress and deformation (Newtonian constitutive equation)
- Microscopic and macroscopic momentum balances
- Internal flows – pipes, conduits
- External flows – drag, boundary layers
- Advanced fluid mechanics – complex shapes

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Momentum . . . is a vector \( \rho \vec{v} = \left( \rho v_x \rho v_y \rho v_z \right)_{xyz} \)

Microscopic momentum balance
\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla P + \mu \nabla^2 \vec{v} + \rho g
\]
Ch6

Macroscopic momentum balance
\[
\frac{d P}{dt} + \sum_{i=1}^{n} \left[ \frac{\rho A \cos \theta \left( \vec{v} \right)^2}{\beta} \right] = \sum_{i=1}^{n} \left[ -pA\hat{n} \right] + R + M_{cv}g
\]
Ch9

So we need vector math.

Vectors
\[
\vec{v} = \left( v_x \right) = \left( v_1 \right) = \left( v_r \right) = \left( \begin{array} {c} v_x \\ v_y \\ v_z \end{array} \right)_{xyz} = \left( \begin{array} {c} v_1 \\ v_2 \\ v_3 \end{array} \right)_{123} = \left( \begin{array} {c} v_r \\ v_\theta \\ v_\phi \end{array} \right)_{r\theta\phi}
\]

Note:
\( v_x \neq v_1 \neq v_r \) (usually)

same vector, different coordinate systems
\[ |\vec{v}| = v = \text{vector magnitude} \]
\[ \frac{\vec{v}}{v} = \hat{v} = \text{unit vector} \]

We will choose coordinate systems for convenience.
Fluid Velocity is a Vector Field

\[ \mathbf{v} = v(x, y, z) \]

The flow is a steady upward flow; the length and direction of the vector indicates the velocity at that location.

Vectors – Cartesian coordinate system

\[ \mathbf{V} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \]

\[ \mathbf{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{xyz} = v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}} + v_3 \hat{\mathbf{k}} \]

- We do algebra with these the same way as with other quantities
- The cartesian basis vectors are constant (with position)
Vectors – Cylindrical coordinate system

\[ \mathbf{V} = \begin{pmatrix} V_r \\ V_\theta \\ V_z \end{pmatrix} \begin{pmatrix} r \theta z \end{pmatrix} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z \]

• The cylindrical basis vectors are variable (with position)

\[
\begin{align*}
x &= r \cos \theta & \hat{e}_r &= \cos \theta \hat{x} + \sin \theta \hat{y} \\
y &= r \sin \theta & \hat{e}_\theta &= -\sin \theta \hat{x} + \cos \theta \hat{y} \\
z &= z & \hat{e}_z &= \hat{z}
\end{align*}
\]

(see inside back cover)

Vectors – Spherical coordinate system

\[ \mathbf{V} = \begin{pmatrix} V_r \\ V_\theta \\ V_\phi \end{pmatrix} \begin{pmatrix} r \theta \phi \end{pmatrix} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_\phi \hat{e}_\phi \]

• The spherical basis vectors are variable (with position)

\[
\begin{align*}
x &= r \sin \theta \cos \phi & \hat{e}_r &= \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \\
y &= r \sin \theta \sin \phi & \hat{e}_\theta &= \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} + (-\sin \theta) \hat{z} \\
z &= r \cos \theta & \hat{e}_\phi &= (-\sin \phi) \hat{x} + \cos \phi \hat{y}
\end{align*}
\]

(see inside back cover)
Fluid Velocity is a Vector Field

Velocity magnitude and direction varies with position

\[ \mathbf{v} = v(x, y, z) \]

Example 1: At positions (1,45°,0) and (1,90°,0) in the r\(\theta\)z coordinate system, the velocity vector of a fluid is given by

\[
\begin{pmatrix}
  v_r \\
  v_\theta \\
  v_z
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  1 \\
  0
\end{pmatrix}
\]

What is this vector in the usual xyz coordinate system?
We use Calculus in Fluid Mechanics to:

1. Calculate flow rate
2. Calculate average velocity
3. Express forces on surfaces due to fluids
4. Express torques on surfaces due to fluids

1. Calculate Flow rate: \( Q \) or \( \dot{V} \)

General: 
\[
Q = \iint_{\text{area}} (\mathbf{v} \cdot \hat{n}) d(\text{area})
\]

Tube flow: 
\[
Q = \int_{0}^{2\pi} \int_{0}^{R} v_z(r) r dr d\theta
\]

\( (\mathbf{v} \cdot \hat{n}) \) is the component of \( \mathbf{v} \) in the direction normal to the area
Common surface shapes:

- **rectangular**: \( d(\text{area}) = dx \, dy \)
- **circular**: \( d(\text{area}) = r \, dr \, d\theta \)
- **surface of cylinder**: \( d(\text{area}) = R \, d\theta \, dz \)
- **spherical**: \( d(\text{area}) = (r \, d\theta) (r \, \sin \theta \, d\phi) = r^2 \sin \theta \, d\theta \, d\phi \)

(see inside back cover)

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**Example 2:** Calculate the flow rate in flow down an incline plane of width \( W \).

Momentum balance calculation gives:

\[
v_z(x) = \frac{\rho g \cos(\beta)}{2\mu} \left( H^2 - x^2 \right)
\]
2. Calculate Average velocity: \( \langle v \rangle \)

**General:**
\[
\langle v \rangle = \frac{Q}{\text{area}}
\]

**Tube flow:**
\[
\langle v \rangle = \frac{Q}{\pi R^2}
\]

“area” is the cross-sectional area normal to flow

---

**Example 3:** The shape of the velocity profile for a steady flow in a tube is found to be given by the function below. Over the range 0 <\( r < 10 \) mm, (\( R=10\) mm), what is the average value of the velocity?

\[
\frac{v_z}{v_{\text{max}}} = f(r) = 1 - \left( \frac{r}{10} \right)^2
\]
3. Express forces on surfaces due to fluids

\[
\text{total fluid force on a surface} = \mathcal{F} = \iint_S \left[ \hat{n} \cdot \Pi \right] \, dS
\]

\[
\Pi = \tau - pI = \text{total stress tensor}
\]

Example 4: In a liquid of density \( \rho \), what is the net fluid force on a submerged sphere (a ball or a balloon)? What is the direction of the force and how does the magnitude of the fluid force vary with fluid density?
Lecture 2 F. Morrison CM3110

Solution: We will be able to do this in this course (Ch4, p257).

From expression for force due to fluid, obtain (spherical coordinates):

\[
\mathcal{F} = \iiint_{\mathcal{S}} \hat{n} \cdot \Pi \, dS
\]

\[
\mathcal{F} = -\rho g R^2 \int_0^{2\pi} \int_0^\pi (H_0 - R \cos \theta) \hat{\theta} \sin \theta \, d\theta \, d\phi
\]

We can do the math from here.

4. Express torques on surfaces due to fluids

\[
\mathcal{T} = \iiint_{\mathcal{S}} \mathbf{R} \times \hat{n} \cdot \Pi \, dS
\]

\[
\mathbf{R} = \text{lever arm}
\]

\[
\Pi = \tau - \rho \mathbf{I} \text{ = total stress tensor}
\]

We will learn to write the stress tensor for our systems; then we can calculate stresses, torques.
Example 5, Torque in Couette Flow: A cup-and-bob apparatus is widely used to measure viscosities for fluids. For the apparatus below, what is the torque needed to turn the inner cylinder (called the bob) at an angular speed of $\Omega$?

Torque in Couette Flow

Solution:

1. Solve for velocity field (microscopic momentum bal)
2. Calculate stress tensor
3. Formulate equation for torque (an integral)
4. Integrate
5. Apply boundary conditions
Torque in Couette Flow
Solution:

Velocity solution:
\[
\mathbf{V} = \begin{pmatrix}
0 \\
\left(\frac{k^2 \Omega R}{k^2 - 1}\right) \left(\frac{r}{R} - \frac{r}{R}ight) \\
0
\end{pmatrix}_{rz}
\]

\[
\tau = \mu \left( \nabla \mathbf{V} + (\nabla \mathbf{V})^T \right)
\]

\[
\Pi = \tau - \rho \mathbf{I}
\]

\[
\text{total fluid torque on a surface} = \oint_S \left[ R \times [\hat{n} \cdot \Pi] \right] dS
\]  

What is lever arm, R?

Etc…

Summary of Quick Start

A: Mechanical Energy Balance
1. SI-SO, steady, incompr, no rxn, no T, no Q
2. Macroscopic
3. Choose points 1 and 2 wisely
4. Solve for \( F \) or \( W_{s, on} \) or \( p \), velocity, elevation

B): Use Calculus in Fluid Mechanics to
1. Calculate flow rate
2. Calculate average velocity
3. Express forces on surfaces due to fluids
4. Express torques on surfaces due to fluids