

CM3110
Transport I
Part II: Heat Transfer

MichiganTech

***One-Dimensional Heat
Transfer - Unsteady***



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Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature T_0 . At time $t = 0$, the left face of the slab is exposed to an environment at temperature T_1 . What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, k , density, ρ , and heat capacity, C_p .

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Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature T_0 . At time $t = 0$, the left face of the slab is **exposed to an environment** at temperature T_1 . What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, k , density, ρ , and heat capacity, C_p .

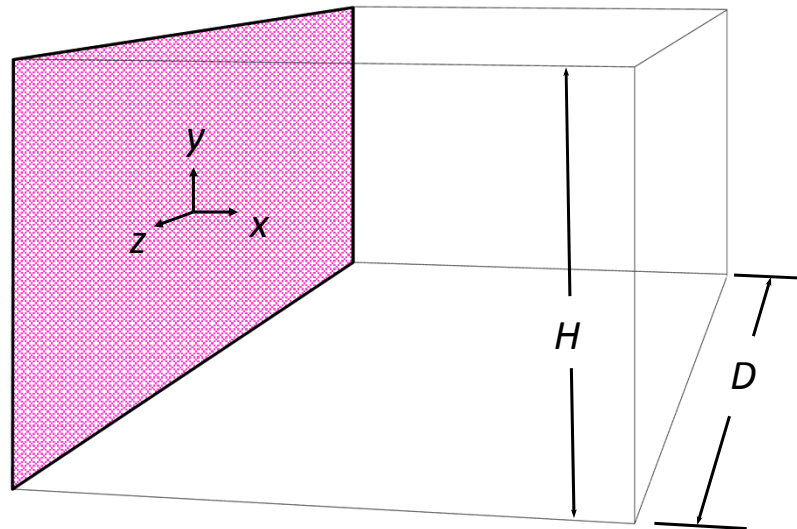
This is code for:

“Newton’s law of cooling boundary conditions:”
 $|\text{flux}| = h|T_{\text{bulk}} - T_{\text{wall}}|$

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Unsteady State Heat Transfer

Example: Unsteady Heat Conduction in a Semi-infinite solid

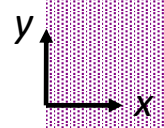


H, D, very large

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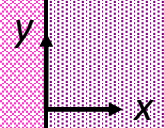
Initial Condition:

$t < 0$
 $T = T_o$



$t < 0$
 $T = T_o$

$t \geq 0$
 $T = T_1$



$t > 0$
 $T = T(x, t)$

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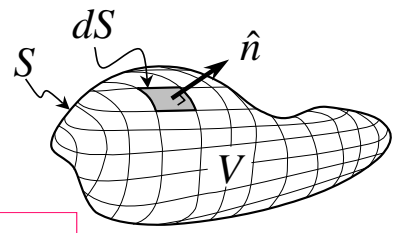
General Energy Transport Equation

(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, V , enclosed by a surface, S .

Gibbs notation:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$



see handout for component notation

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General Energy Transport Equation

(microscopic energy balance)

$$\underbrace{\rho \hat{C}_p}_{\text{rate of change}} \left(\underbrace{\frac{\partial T}{\partial t} + \underbrace{\mathbf{v} \cdot \nabla T}_{\text{convection}} \right) = \underbrace{k \nabla^2 T}_{\text{conduction (all directions)}} + \underbrace{S}_{\text{source (energy generated per unit volume per time)}}$$

velocity must satisfy equation of motion, equation of continuity

see handout for component notation

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Equation of energy

for Newtonian fluids of constant density, ρ , and thermal conductivity, k , with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

CM310 Fall 1999 Faith Morrison

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

Note: this handout is on the web:
www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \frac{k}{\rho \hat{C}_p} \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

Cylindrical (r θ z) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

Spherical (r θ ϕ) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} = \frac{k}{\rho \hat{C}_p} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + \frac{S}{\rho \hat{C}_p}$$

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Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature T_0 . At time $t = 0$, the left face of the slab is exposed to an environment at temperature T_1 . What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, k , density, ρ , and heat capacity, C_p .

Newton's law of cooling BC's:

$$|q_x| = hA|T_{bulk} - T_{surface}|$$

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Microscopic Energy Equation in Cartesian Coordinates

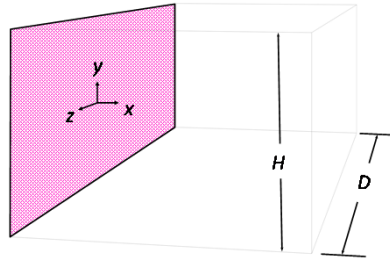
$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} = \text{thermal diffusivity}$$

what are the boundary conditions? initial conditions?

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Example 7: Unsteady Heat Conduction in a Semi-infinite solid



Initial Condition:

$$t < 0 \quad T = T_o \quad \text{for } x \geq 0$$

$$t \geq 0 \quad T = T_1 \quad \text{for } x = 0$$

$$t > 0 \quad T = T(x, t) \quad \text{for } x > 0$$

You try.

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Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} \right) = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

Initial condition: $t = 0, T = T_o \quad \forall x$

Boundary conditions:

$$x = 0, \quad q_x = hA(T - T_1) \quad \forall t > 0$$

$$x = \infty, \quad T = T_o \quad \forall t$$

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Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} \right) = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

The solution is obtained by combination of variables.

Initial condition: $t = 0, T = T_0 \forall x$

Boundary conditions:

$$x = 0, q_x = hA(T - T_1) \quad \forall t > 0$$

$$x = \infty, T = T_0 \quad \forall t$$

$$\beta \equiv \frac{h\sqrt{\alpha t}}{k}$$

$$\zeta \equiv \frac{x}{2\sqrt{\alpha t}}$$

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Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\frac{T - T_0}{T_1 - T_0} = \text{erfc } \zeta - e^{\beta(2\zeta + \beta)} \text{erfc}(\zeta + \beta)$$

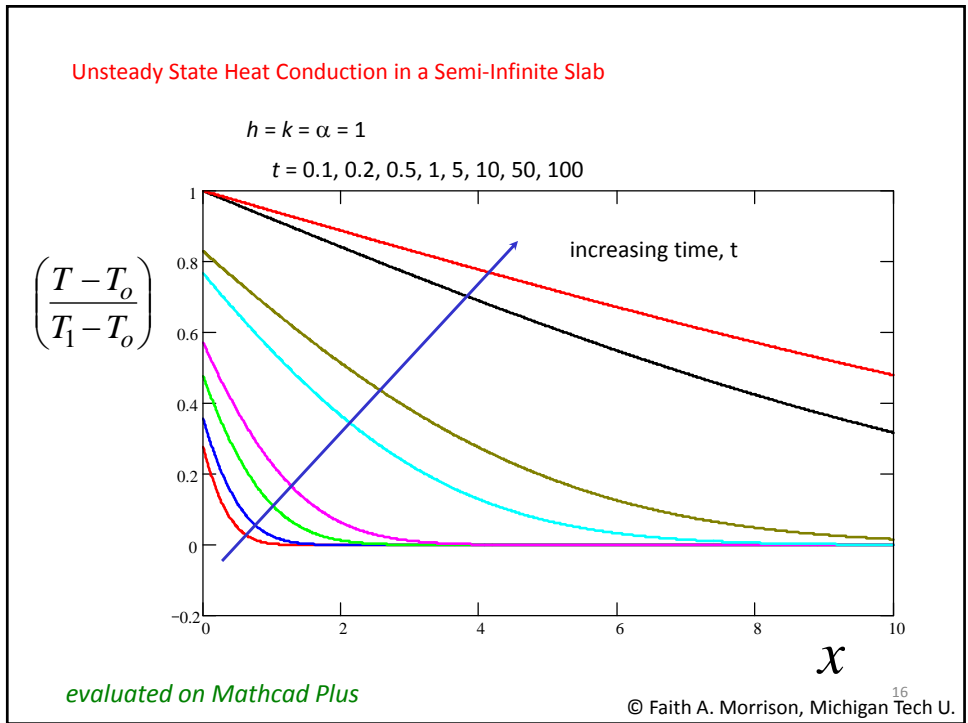
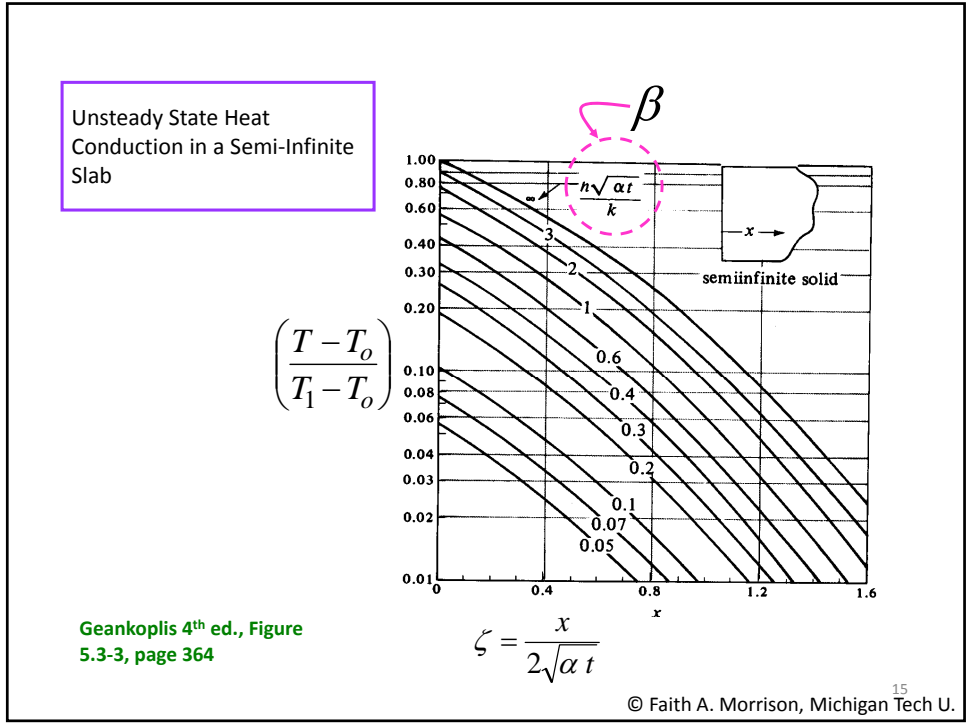
$$\beta \equiv \frac{h\sqrt{\alpha t}}{k} \quad \zeta \equiv \frac{x}{2\sqrt{\alpha t}}$$

Geankoplis 4th ed., eqn 5.3-7, page 363

complementary error function of y $\text{erfc}(y) \equiv 1 - \text{erf}(y)$

error function of y $\text{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-(y')^2} dy'$

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How could we use this solution?
 Example: Will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 ft under ground? Use the following physical properties:

$$h = 2.0 \frac{BTU}{h \text{ ft}^2 \text{ } ^\circ F}$$

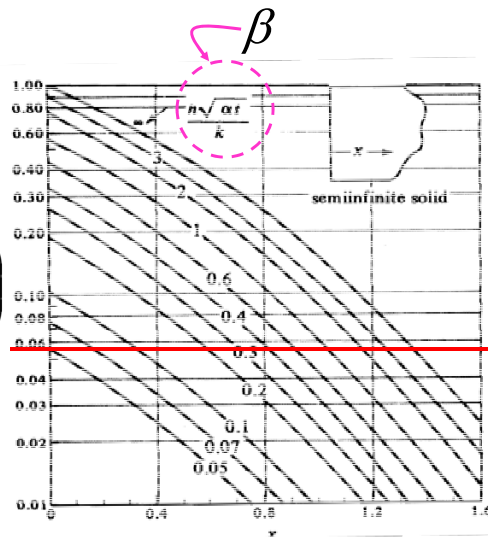
$$\alpha_{soil} = 0.018 \frac{ft^2}{h}$$

$$k_{soil} = 0.5 \frac{BTU}{h \text{ ft } ^\circ F}$$

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Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\left(\frac{T - T_o}{T_1 - T_o} \right)$$



$$\zeta = \frac{x}{2\sqrt{\alpha t}}$$

Geankoplis 4th ed., Figure 5.3-3, page 364

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Example 8: Unsteady Heat Conduction in a Finite-sized solid

- The slab is tall and wide, but of thickness $2H$
- Initially at T_0
- at time $t = 0$ the temperature of the sides is changed to T_1

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Unsteady State Heat Transfer

Use same microscopic energy balance eqn as before.

$$\underbrace{\rho \hat{C}_p}_{\text{rate of change}} \left(\underbrace{\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T}_{\text{convection}} \right) = \underbrace{k \nabla^2 T}_{\text{conduction (all directions)}} + \underbrace{S}_{\text{source (energy generated per unit volume per time)}}$$

see handout for component notation

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Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

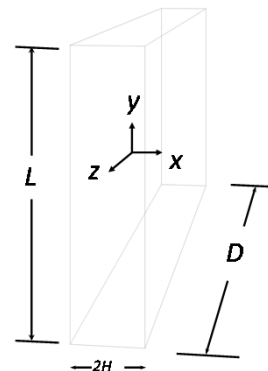
$$\alpha \equiv \frac{k}{\rho \hat{C}_p} = \text{thermal diffusivity}$$

what are the boundary conditions? initial conditions?

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Example 8: Unsteady Heat Conduction in a Finite-sized solid

You try.



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Unsteady State Heat Conduction in a Finite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} \right) = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

Initial condition: $t = 0, T = T_o \forall x$

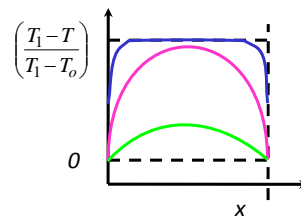
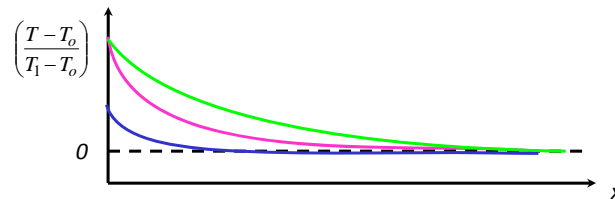
Boundary conditions:

$$\left. \begin{array}{l} x = 0, \quad T = T_1 \\ x = 2H, \quad T = T_1 \end{array} \right\} \forall t > 0$$

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Q: How can two completely different situations give the same governing equation?

A: The boundary conditions make all the difference



For more solutions to this equation see Carslaw and Jaeger, *Conduction of Heat in Solids*, 2nd Edition, Oxford, 1959

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Unsteady State Heat Conduction in a Finite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} \right) = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

The solution is obtained by separation of variables.

Initial condition: $t = 0, T = T_o \forall x$

Boundary conditions:

$$\left. \begin{array}{l} x = 0, \quad T = T_1 \\ x = 2H, \quad T = T_1 \end{array} \right\} \forall t > 0$$

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Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

$$\text{Let } Y \equiv \left(\frac{T_1 - T}{T_1 - T_o} \right) \quad \frac{\partial Y}{\partial t} = \alpha \left(\frac{\partial^2 Y}{\partial x^2} \right)$$

Guess: $Y = X(x)\Theta(t)$

Initial condition:

$$t = 0, T = T_o \forall x \Rightarrow Y = 1$$

Boundary conditions:

$$\left. \begin{array}{l} x = 0, \quad T = T_1 \Rightarrow Y = 0 \\ x = 2H, \quad T = T_1 \Rightarrow Y = 0 \end{array} \right\} \forall t > 0$$

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Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$$Y = X(x)\Theta(t) \qquad \frac{\partial Y}{\partial t} = \alpha \left(\frac{\partial^2 Y}{\partial x^2} \right)$$

$$\frac{\partial Y}{\partial t} = \frac{\partial}{\partial t} (X(x)\Theta(t)) = X(x) \frac{d\Theta(t)}{dt}$$

$$\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x} (X(x)\Theta(t)) = \frac{dX(x)}{dx} \Theta(t)$$

$$\frac{\partial^2 Y}{\partial x^2} = \frac{d^2 X(x)}{dx^2} \Theta(t)$$

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Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$$\frac{\partial Y}{\partial t} = \alpha \left(\frac{\partial^2 Y}{\partial x^2} \right)$$

Substituting:

$$X(x) \frac{d\Theta(t)}{dt} = \alpha \frac{d^2 X(x)}{dx^2} \Theta(t)$$

$$\underbrace{\frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt}}_{\text{function of time (t) only}} = \alpha \underbrace{\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}}_{\text{function of position (x) only}} \Rightarrow = \lambda$$

constant

The function of two variables is separable into two functions of one variable.

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Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

Separates into two ordinary differential equations:

$$\frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt} = \lambda$$

$$\alpha \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \lambda$$

The function of two variables is separable into two functions of one variable.

Solve.

Apply BCs.

Apply ICs.

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Temperature Profile for Unsteady State Heat Conduction in a Finite Slab

$$\left(\frac{T_1 - T}{T_1 - T_o} \right) = \frac{4}{\pi} \left\{ e^{-\frac{\pi^2 \alpha t}{4H^2}} \sin \frac{\pi x}{2H} + \frac{1}{3} e^{-\frac{3^2 \pi^2 \alpha t}{4H^2}} \sin \frac{3\pi x}{2H} + \frac{1}{5} e^{-\frac{5^2 \pi^2 \alpha t}{4H^2}} \sin \frac{5\pi x}{2H} + \dots \right\}$$

Geankoplis 4th ed., eqn 5.3-6, p363

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Microscopic Energy Balance – is the correct physics for many problems!

Tricky step:

solving for T field; this can be mathematically difficult

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- may not be separable from mass and momentum balances

Strategy: solve using numerical methods
 (e.g. *Comsol*)
 **** Or ****
 Develop correlations on complex systems by using *Dimensional Analysis*

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**CM3110
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More Complex Heat Transfer – Dimensional Analysis

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