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Coordinate system	surface differential dS
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dxdy$
Cartesian (side a, $\hat{n} = \hat{e}_y$)	$dS = dxdz$
Cartesian (side b, $\hat{n} = \hat{e}_x$)	$dS = dydz$
cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = rdrd\theta$
cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = Rd\theta dz$
spherical, ($\hat{n} = \hat{e}_r$)	$dS = R^2 \sin\theta d\theta d\phi$

What are $\hat{e}_x, \hat{e}_y, \hat{e}_z$ in terms of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$?

SPHERICAL COORDS

Coordinate system	volume differential dV
Cartesian	$dV = dxdydz$
cylindrical	$dV = rdrd\theta dz$
spherical	$dV = r^2 \sin\theta dr d\theta d\phi$

Coordinate system	coordinates	basis vectors
spherical	$x = r\sin\theta\cos\phi$	$\hat{e}_r = (\sin\theta\cos\phi\hat{e}_x) + (\sin\theta\sin\phi\hat{e}_y) + \cos\theta\hat{e}_z$
	$y = r\sin\theta\sin\phi$	$\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z$
	$z = r\cos\theta$	$\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\hat{e}_y$
cylindrical	$x = r\cos\theta$	$\hat{e}_r = \cos\theta\hat{e}_x + \sin\theta\hat{e}_y$
	$y = r\sin\theta$	$\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\hat{e}_y$
	$z = z$	$\hat{e}_z = \hat{e}_z$

Divergence Theorem $$\iint_S \hat{n}\cdot\underline{F}\, dS = \iiint_V \nabla\cdot\underline{F}\, dV$$

Stokes Theorem $$\oint_C \hat{t}\cdot\underline{F}\, dl = \iint_S \hat{n}\cdot(\nabla\times\underline{F})\, dS$$

Vector identities:

$$\begin{aligned}
\nabla\cdot\nabla\times\underline{F} &= 0 \quad \text{(Divergence of curl = 0)}\\
\nabla\times\nabla f &= 0 \quad \text{(Curl of gradient = 0)}\\
\nabla(fg) &= f\nabla g + g\nabla f\\
\underline{F}\cdot\nabla\underline{F} &= \frac{1}{2}\nabla\left(\underline{F}^2\right) - \underline{F}\times(\nabla\times\underline{F})\\
\nabla\cdot(f\underline{F}) &= f\nabla\cdot\underline{F} + \underline{F}\cdot\nabla f\\
\nabla\times\nabla\times\underline{F} &= \nabla(\nabla\cdot\underline{F}) - \nabla^2\underline{F}\\
\nabla\cdot(\underline{F}\times\underline{G}) &= \underline{G}\cdot(\nabla\times\underline{F}) - \underline{F}\cdot(\nabla\times\underline{G})
\end{aligned}$$

FROM inside back cover:

$$\hat{e}_r = \sin\theta\cos\phi\, e_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

multiply by $\cos\theta$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z$$

multiply by $-\sin\theta$

$$\cos\theta\,\hat{e}_r = \sin\theta\cos\phi\cos\theta\,\hat{e}_x + \sin\theta\cos\theta\sin\phi\,\hat{e}_y + \cos^2\theta\,\hat{e}_z$$

$$-\sin\theta\,\hat{e}_\theta = -\sin\theta\cos\phi\cos\theta\,\hat{e}_x - \sin\theta\cos\theta\sin\phi\,\hat{e}_y + \sin^2\theta\,\hat{e}_z$$

ADD:

$$\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = \overbrace{(\cos^2\theta + \sin^2\theta)}^{1}\, e_z$$

$$\boxed{\hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta}$$

Substitute $\hat{e}_z$ back into $\hat{e}_r$ expression:

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos^2\theta\,\hat{e}_r - \sin\theta\cos\theta\,\hat{e}_\theta$$

re-arrange:

$$\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}\hat{e}_r = \cancel{\sin}\theta\cos\phi\,\hat{e}_x + \cancel{\sin}\theta\sin\phi\,\hat{e}_y - \cancel{\sin}\theta\cos\theta\,\hat{e}_\theta$$

$\Longrightarrow$

$$\sin\theta\,\hat{e}_r = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y - \cos\theta\,\hat{e}_\theta$$

Now, solve for $\hat{e}_y$:

$$\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = \cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y$$

multiply by $\sin\phi$

$$\hat{e}_\phi = \underbrace{-\sin\phi\,\hat{e}_x}_{\hat{e}_x \text{ terms cancel}} + \cos\phi\,\hat{e}_y$$

multiply by $\cos\phi$

ADD

$$\sin\phi\sin\theta\,\hat{e}_r + \sin\phi\cos\theta\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi = \underbrace{(\sin^2\phi + \cos^2\phi)}_{1}\hat{e}_y$$

$$\boxed{\hat{e}_y = \sin\phi\sin\theta\,\hat{e}_r + \sin\phi\cos\theta\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi}$$

Now, solve for $\hat{e}_x$:

Begining w/ bracketed equations:

1. multiply first by $\cos\phi$
2. multiply second by $-\sin\phi$
3. add

$\Rightarrow$

$$\left.\begin{aligned} \cos\phi\sin\theta\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta &= \cos^2\phi\,\hat{e}_x + \sin\phi\cos\phi\,\hat{e}_y \\ -\sin\phi\,\hat{e}_\phi &= \underbrace{\sin^2\phi\,\hat{e}_x}_{\sin^2\phi+\cos^2\phi=1} - \sin\phi\cos\phi\,\hat{e}_y \end{aligned}\right\} \text{add}$$

cancel

$\Rightarrow$

$$\boxed{\hat{e}_x = \cos\phi\sin\theta\,\hat{e}_r + \cos\phi\cos\theta\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi}$$