Equations Summary from Inside Cover of Morrison, 2013

Mechanical Energy Balance

$$\frac{\Delta p}{\rho} + \frac{\Delta \langle v \rangle^2}{2\alpha} + g\Delta z + F_{friction} = -\frac{W_{s,by\ fluid}}{m} \qquad \begin{cases} \alpha_{laminar} = 0.5 \\ \alpha_{turbulent} \approx 1 \end{cases}$$

$$F_{friction} = \left[4f\frac{L}{D} + \sum_{fittings_i} n_i K_{f,i} \right] \frac{\langle v \rangle^2}{2}$$

Fanning Friction Factor (pipe flow)

$$f = \frac{F_{drag}}{\frac{1}{2}\rho\langle v \rangle^2 \, (2\pi RL)} = \frac{\Delta p D}{2L\rho\langle v \rangle^2}$$

Note this is correct; there is an error on the inside cover

Drag Coefficient (sphere drop)

$$C_D = \frac{F_{drag}}{\frac{1}{2}\rho v_\infty^2 \, (\pi R^2)} = \frac{4gD(\rho_{body} - \rho)}{3\rho v_\infty^2}$$

Momentum balance on a CV (Reynolds transport theorem)

$$\frac{d\underline{\mathbf{P}}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v})\rho\underline{v}\, dS = \sum_{on\ CV} \underline{f}$$

Hydrostatic Pressure

$$p_{bottom} = p_{top} + \rho g h$$

Hagen-Poiseuille Equation (steady, laminar tube flow, incompressible)

$$Q = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$$

Prandtl Equation (steady, turbulent tube flow)

$$\frac{1}{\sqrt{f}} = -4.0 \log\left(\frac{4.67}{\mathrm{Re}\sqrt{f}}\right) + 2.28$$

Stokes-Einstein-Sutherland Equation (steady, slow flow around a sphere)

$$F_{drag} = 6\pi R\mu v_\infty$$

Macroscopic Momentum Balance on a CV

$$\frac{d\mathbf{P}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos(\theta)\langle v \rangle^2}{\beta}\hat{v} \right]\Bigg|_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g} \qquad \begin{cases} \beta_{laminar} = 0.75 \\ \beta_{turbulent} \approx 1 \end{cases}$$

Navier-Stokes equation (microscopic momentum balance, incompressible, Newtonian fluids)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Continuity equation (microscopic mass balance, incompressible fluids)

$$\nabla \cdot \underline{v} = 0$$

Total stress tensor $\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$

$$\begin{pmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\ \tilde{\Pi}_{31} & \tilde{\Pi}_{32} & \tilde{\Pi}_{33} \end{pmatrix}_{123} = \begin{pmatrix} \tilde{\tau}_{11} - p & \tilde{\tau}_{12} & \tilde{\tau}_{13} \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} - p & \tilde{\tau}_{23} \\ \tilde{\tau}_{31} & \tilde{\tau}_{32} & \tilde{\tau}_{33} - p \end{pmatrix}_{123}$$

Dynamic pressure $\mathcal{P} \equiv p + \rho g h$

Newtonian constitutive equation

$$\begin{aligned} \underline{\underline{\tilde{\tau}}} &= \mu \left(\nabla \underline{v} + (\nabla \underline{v})^T \right) \\ &= \mu \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123} \end{aligned}$$

Total molecular fluid force on a finite surface $\mathcal{S}$

$$\underline{\mathcal{F}} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{\text{at surface}} dS$$

Stationary fluid $\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right] = -p\hat{n}$

Moving fluid $\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right] = -p\hat{n} + \hat{n} \cdot \underline{\underline{\tilde{\tau}}}$

Total fluid torque on a finite surface $\mathcal{S}$

$$\underline{\mathcal{T}} = \iint_{\mathcal{S}} \left[\underline{R} \times \left(\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right) \right]_{\text{at surface}} dS$$

Total flow rate out through a finite surface $\mathcal{S}$

$$Q = \dot{V} = \iint_{\mathcal{S}} [\hat{n} \cdot \underline{v}]_{\text{at surface}} \; dS$$

Average velocity across a finite surface $\mathcal{S}$

$$\langle v \rangle = \frac{Q}{\mathcal{S}}$$

Coordinate system	surface differential dS
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dxdy$
Cartesian (side a, $\hat{n} = \hat{e}_y$)	$dS = dxdz$
Cartesian (side b, $\hat{n} = \hat{e}_x$)	$dS = dydz$
cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = rdrd\theta$
cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = Rd\theta dz$
spherical, ($\hat{n} = \hat{e}_r$)	$dS = R^2 \sin\theta d\theta d\phi$

Coordinate system	volume differential dV
Cartesian	$dV = dxdydz$
cylindrical	$dV = rdrd\theta dz$
spherical	$dV = r^2 \sin\theta drd\theta d\phi$

Coordinate system	coordinates	basis vectors
spherical	$x = r\sin\theta\cos\phi$	$\hat{e}_r = (\sin\theta\cos\phi\hat{e}_x) + (\sin\theta\sin\phi\hat{e}_y) + \cos\theta\hat{e}_z$
	$y = r\sin\theta\sin\phi$	$\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z$
	$z = r\cos\theta$	$\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\hat{e}_y$
cylindrical	$x = r\cos\theta$	$\hat{e}_r = \cos\theta\hat{e}_x + \sin\theta\hat{e}_y$
	$y = r\sin\theta$	$\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\hat{e}_y$
	$z = z$	$\hat{e}_z = \hat{e}_z$

$$\text{Divergence Theorem} \qquad \iint_{\mathcal{S}} \hat{n}\cdot\underline{F}\; dS = \iiint_{\mathcal{V}} \nabla\cdot\underline{F}\; dV$$

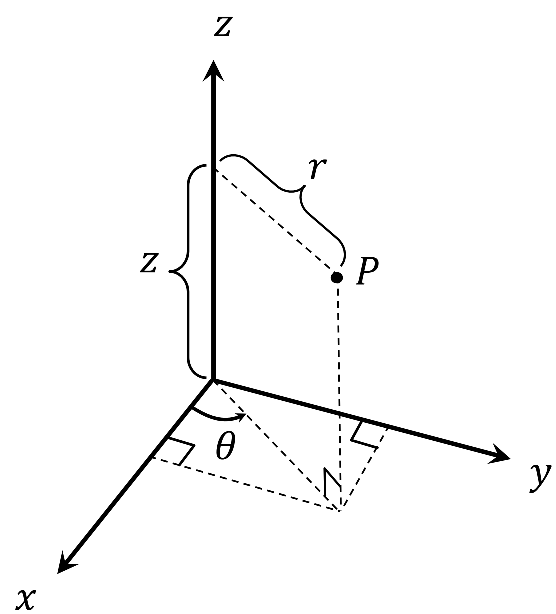
$$\text{Stokes Theorem} \qquad \oint_{\mathcal{C}} \hat{t}\cdot\underline{F}\; dl = \iint_{\mathcal{S}} \hat{n}\cdot\left(\nabla\times\underline{F}\right)\; dS$$

Vector identities:

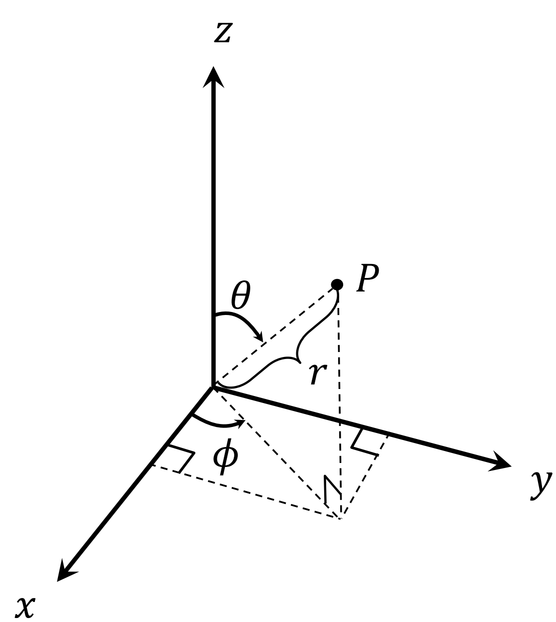
$$\begin{aligned}
\nabla\cdot\nabla\times\underline{F} &= 0 \quad \text{(Divergence of curl} = 0\text{)} \\
\nabla\times\nabla f &= 0 \quad \text{(Curl of gradient} = 0\text{)} \\
\nabla\left(fg\right) &= f\nabla g + g\nabla f \\
\underline{F}\cdot\nabla\underline{F} &= \frac{1}{2}\nabla\left(\underline{F}^2\right) - \underline{F}\times\left(\nabla\times\underline{F}\right) \\
\nabla\cdot\left(f\underline{F}\right) &= f\nabla\cdot\underline{F} + \underline{F}\cdot\nabla f \\
\nabla\times\nabla\times\underline{F} &= \nabla\left(\nabla\cdot\underline{F}\right) - \nabla^2\underline{F} \\
\nabla\cdot\left(\underline{F}\times\underline{G}\right) &= \underline{G}\cdot\left(\nabla\times\underline{F}\right) - \underline{F}\cdot\left(\nabla\times\underline{G}\right)
\end{aligned}$$

The equations in F. A. Morrison, ***An Introduction to Fluid Mechanics*** (Cambridge, 2013) assume the following definitions of the cylindrical and spherical coordinate systems.

Cylindrical Coordinate System: Note that the θ-coordinate swings around the z-axis



Spherical Coordinate System: Note that the θ-coordinate swings down from the z-axis; this is different from its definition in the cylindrical system above.



The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned}
\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x \\
\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y \\
\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z
\end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\begin{aligned}
\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r} \frac{\partial (r \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z} \right) + \rho g_r \\
\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2} \frac{\partial (r^2 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} \right) + \rho g_\theta \\
\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{1}{r} \frac{\partial (r \tilde{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z
\end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right)$$

$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2} \frac{\partial (r^2 \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tilde{\tau}_{\theta r} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right)$$

$$= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3} \frac{\partial (r^3 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tilde{\tau}_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi} \cot \theta}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right)$$

$$= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3} \frac{\partial (r^3 \tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \left(\tilde{\tau}_{\theta\phi} \sin \theta \right)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\phi$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\begin{aligned}
\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) &= -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x \\
\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) &= -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y \\
\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z
\end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\begin{aligned}
\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) &= -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r \\
\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right) &= -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta \\
\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z
\end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\begin{aligned}
&\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right) \\
&\quad = -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right. \\
&\qquad \left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r
\end{aligned}$$

$$\begin{aligned}
&\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right) \\
&\quad = -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right. \\
&\qquad \left.+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta
\end{aligned}$$

$$\begin{aligned}
&\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right) \\
&\quad = -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right. \\
&\qquad \left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi
\end{aligned}$$

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2^{nd} edition, Wiley: NY, 2002.

2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

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FACTORS FOR UNIT CONVERSIONS

Quantity	Equivalent Values
Mass	1 kg = 1000 g = 0.001 metric ton = 2.20462 lb_m = 35.27392 oz 1 lb_m = 16 oz = 5 x 10^{-4} ton = 453.593 g = 0.453593 kg
Length	1 m = 100 cm = 1000 mm = 10^6 microns (μm) = 10^{10} angstroms (Å) = 39.37 in = 3.2808 ft = 1.0936 yd = 0.0006214 mile 1 ft = 12 in. = 1/3 yd = 0.3048 m = 30.48 cm
Volume	1 m^3 = 1000 liters = 10^6 cm^3 = 10^6 ml = 35.3145 ft^3 = 220.83 imperial gallons = 264.17 gal = 1056.68 qt 1 ft^3 = 1728 in^3 = 7.4805 gal = 0.028317 m^3 = 28.317 liters = 28 317 cm^3
Force	1 N = 1 $kg \cdot m/s^2$ = 10^5 dynes = 10^5 $g \cdot cm/s^2$ = 0.22481 lb_f 1 lb_f = 32.174 $lb_m \cdot ft/s^2$ = 4.4482 N = 4.4482 x 10^5 dynes
Pressure	1 atm = 1.01325 x 10^5 N/m^2 (Pa) = 101.325 kPa = 1.01325 bars = 1.01325 x 10^6 $dynes/cm^2$ = 760 mm Hg at 0˚ C (torr) = 10.333 m H_2O at 4˚ C = 14.696 lb_f/in^2 (psi) = 33.9 ft H_2O at 4˚C 100 kPa = 1 bar
Energy	1 J = 1 $N \cdot m$ = 10^7 ergs = 10^7 $dyne \cdot cm$ = 2.778 x 10^{-7} $kW \cdot h$ = 0.23901 cal = 0.7376 $ft \cdot lb_f$ = 9.47817 x 10^{-4} Btu
Power	1 W = 1 J/s = 0.23885 cal/s = 0.7376 $ft \cdot lb_f/s$ = 9.47817 x 10^{-4} Btu/s = 3.4121 Btu/h = 1.341 x 10^{-3} hp (horsepower)
Viscosity	1 $Pa \cdot s$ = 1 $N \cdot s/m^2$ = 1 $kg/m \cdot s$ = 10 poise = 10 $dynes \cdot s/cm^2$ = 10 $g/cm \cdot s$ = 10^3 cp (centipoise) = 0.67197 $lb_m/ft \cdot s$ = 2419.088 $lb_m/ft \cdot h$
Density	1 kg/m^3 = 10^{-3} g/cm^3 = 0.06243 lb_m/ft^3 10^3 kg/m^3 = 1 g/cm^3 = 62.428 lb_m/ft^3
Volumetric Flow	1 m^3/s= 35.3145 ft^3/s=15,850.2 gal/min (gpm) 1 gpm = 6.30907 x 10^{-5} m^3/s=2.22802 x 10^{-3} ft^3/s=3.7854 liter/min 1 liter/min=0.26417 gpm

Temperature $T(^oC) = \frac{5}{9}\left[T(^oF) - 32\right]$

$T(^oF) = \frac{9}{5}T(^oC) + 32 = 1.8T(^oC) + 32$

Absolute Temperature T(K) = T(˚C) + 273.15

T(˚R) = T(˚F) + 459.67

Temperature Interval (ΔT) 1 C˚ = 1 K = 1.8 F˚ = 1.8 R˚

1F˚ = 1 R˚ = (5/9) C˚ = (5/9) K

USEFUL QUANTITIES

SG = $\rho(20^\circ C)/\rho_{water}(4^\circ C)$

$\rho_{water}(4^\circ C)$ = 1000 kg/m^3 = 62.43 lb$_m$/ft^3 = 1.000 g/cm^3

$\rho_{water}(25^\circ C)$ = 997.08 kg/m^3 = 62.25 lb$_m$/ft^3 = 0.99709 g/cm^3

g = 9.8066 m/s^2 = 980.66 cm/s^2 = 32.174 ft/s^2

$\mu_{water}(25^\circ C)$ = 8.937 x 10^{-4} Pa·s = 8.937 x 10^{-4} kg/m·s

= 0.8937 cp =0.8937 x 10^{-2} g/cm·s = 6.005 x 10^{-4} lb$_m$/ft·s

Composition of air:

N_2	78.03%
O_2	20.99%
Ar	0.94%
CO_2	0.03%
H_2, He, Ne, Kr, Xe	0.01%
	100.00%

M_{air} = 29 g/mol = 29 kg/kmol = 29 lb$_m$/lbmole

$\hat{C}_{p,water}(25^\circ C)$ = 4.182 kJ/kg K = 0.9989 cal/g˚C = 0.9997 Btu/lb$_m$˚F

R = 8.314 m^3·Pa/mol·K = 0.08314 liter·bar/mol·K = 0.08206 liter·atm/mol·K

= 62.36 liter·mm Hg/mol·K = 0.7302 ft^3·atm/lbmole·˚R

= 10.73 ft^3·psia/lbmole·˚R

= 8.314 J/mol·K

= 1.987 cal/mol·K = 1.987 Btu/lbmole·˚R

Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
Professor Faith A. Morrison

I. Flow through Smooth Pipes

A. All Reynolds numbers: Morrison

The correlation from Morrison (2013) fits the smooth pipe data for all Reynolds numbers; beyond $Re = 4000$ this correlation follows the Prandtl equation (see Figure 1; Morrison, equation 7.158). This correlation is explicit in f; when flow rate is known, Δp may be found directly; when Δp is known, Q or $\langle v \rangle$ must be solved for iteratively.

Morrison (2013)

$$f = \left(\frac{0.0076 \left(\frac{3170}{Re}\right)^{0.165}}{1 + \left(\frac{3170}{Re}\right)^{7.0}} \right) + \frac{16}{Re} \qquad (1)$$

B. $4,000 \leq Re \leq 1 \times 10^6$: Prandtl

The Prandtl correlation for $f(Re)$ in turbulent flow is not explicit in friction factor and must be solved iteratively except when f is known (Morrison, equation 7.156). This is good only for $Re > 4,000/$

Prandtl
or VonKarman-Nikuradse
(Denn, 1980)

$$\frac{1}{\sqrt{f}} = 4.0 \log\left(Re\sqrt{f}\right) - 0.40 \qquad (2)$$

C. $4,000 \leq Re \leq 1 \times 10^6$: A simplified Correlation

For the turbulent regime, an approximate correlation that is much simpler to work with (with a calculator on an exam, for example) is given here and shown in Figure 2 (Morrison, equation 7.157). This is good only for $Re > 4,000$.

Simplified Turbulent
(White, 1974)

$$f = \frac{1.02}{4} (\log Re)^{-2.5} \qquad (3)$$

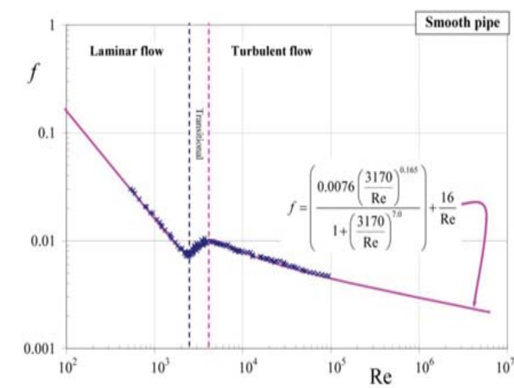

Figure 1: Equation 3 captures smooth pipe friction factor as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are Nikuradse's experimental data for flow in smooth pipes (Nikuradse, 1933). Use beyond $Re = 10^6$ is not recommended; for Re>4000 equation 3 follows the Prandtl equation. (Morrison, 2013, p532)

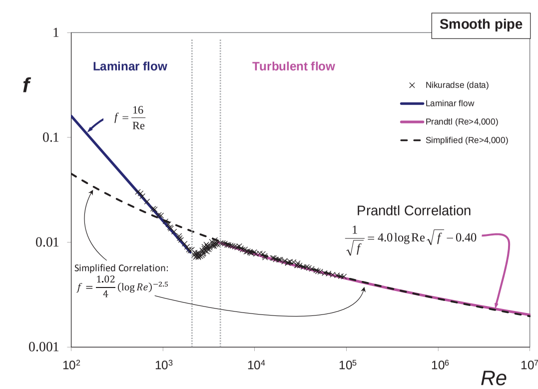

Figure 2: For turbulent flow, the simplified (equation 3) or Prandtl (equation 2) correlations may be used. For work with a calculator, the simplified correlation is perhaps the easiest to work with. (Morrison, 2013, p531)

II. Flow Around a Sphere

A. All Reynolds Numbers: Morrison

The correlation from Morrison (2013) fits the flow around a sphere for all Reynolds numbers (Figure 3; Morrison equation 8.83); beyond $Re = 10^6$ this correlation follows the curve shown in Figure 3.

Morrison (2013)

$$C_D = \frac{24}{Re} + \frac{2.6\left(\frac{Re}{5.0}\right)}{1+\left(\frac{Re}{5.0}\right)^{1.52}} + \frac{0.411\left(\frac{Re}{263{,}000}\right)^{-7.94}}{1+\left(\frac{Re}{263{,}000}\right)^{-8.00}} + \frac{0.25\left(\frac{Re}{10^6}\right)}{1+\left(\frac{Re}{10^6}\right)} \tag{4}$$

Simplified Correlations

The correlations below (Morrison, 2013; equation 8.82) are simpler relationships more suitable to calculator/exam work.

$Re < 2$
$$C_D = \frac{24}{Re} \tag{5}$$

$0.1 \le Re \le 1{,}000$
$$C_D = \frac{24}{Re}(1 + 0.14 Re^{0.7}) \tag{6}$$

$1{,}000 \le Re \le 2.6 \times 10^5$
$$C_D = 0.445 \tag{7}$$

$2.8 \times 10^5 \le Re \le 10^6$
$$\frac{\log C_D}{\left(\frac{Re}{10^6}\right)} = 4.43 \log Re - 27.3 \tag{8}$$

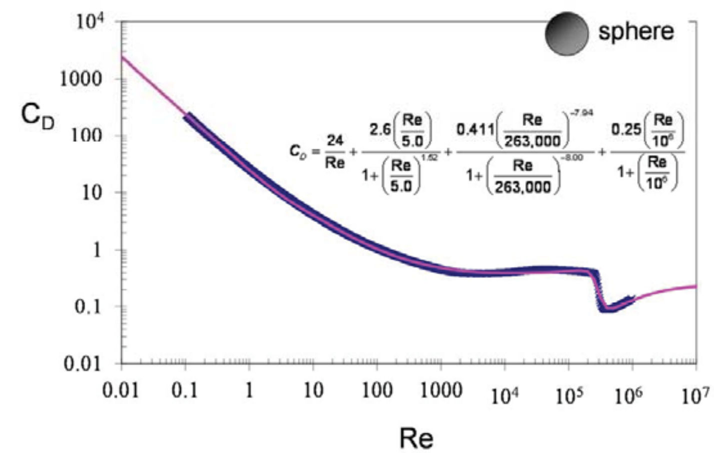

Figure 3: Equation 4 captures flow around a sphere as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are experimental data from White (1974). Use beyond $Re = 10^6$ is not recommended. (Morrison, 2013, p625)

References

M. Denn, Process Fluid Mechanics (Prentice Hall, Englewood Cliffs, NJ, 1980)
F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press, New York, 2013).
F. M. White, *Viscous Fluid Flow* (McGraw-Hill, Inc.: New York, 1974).

The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Prof. Faith A. Morrison, Michigan Technological University

Cartesian Coordinates

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tau_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

Cylindrical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tau_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

Spherical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tau_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi}$$

$$= \mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} & 2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right) \end{pmatrix}_{r\theta\phi}$$

These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

16 April 2014

$$\text{hydraulic diameter (general), } D_H \equiv \frac{4A_{xs}}{\mathcal{p}}$$

$$\text{Po} = \text{Re}_{D_H} f_{D_H}$$

$$\frac{1}{\sqrt{f_{D_H}}} = 4.0 \log\left(\frac{\text{Re}_{D_H}\sqrt{f_{D_H}}}{\frac{\text{Po}_{\text{duct}}}{16}}\right) - 0.40$$

Geometry	Po
Circle	16
Equilateral triangle	13.33
Slit	24
Ellipse (a,b)	$\frac{32\pi^2}{\mathcal{p}}(a^2+b^2)$

$$\text{void fraction, } \varepsilon = \frac{\text{empty bed volume}}{\text{total bed volume}}$$

$$a_v = \frac{\text{total partical surface area}}{\text{particle volume}}$$

$$\text{superficial velocity, } v_0 = \frac{Q}{\mathcal{V}/L}$$

$$\text{hydraulic diameter for packed bed, } D_H = \frac{4\varepsilon}{(1-\varepsilon)a_v}$$

$$\text{Reynolds number for packed bed } = \frac{\rho(v_0/\varepsilon)D_H}{\mu}$$

$$\text{friction factor for packed bed} = \left(\frac{\Delta p}{L}\right)\left(\frac{D_H \varepsilon^2}{2\rho v_0^2}\right)$$

Mechanism	$h, \frac{BTU}{hr\ ft^{2\,o}F}$	$h, \frac{W}{m^2K}$
Condensing steam	1000-5000	5700-28,000
Condensing organics	200-500	1100-2800
Boiling liquids	300-5000	1700-28,000
Moving water	50-3000	280-17,000
Moving hydrocarbons	10-300	55-1700
Still air	0.5-4	2.8-23
Moving air	2-10	11.3-55

Reference: C. J. Geankoplis, Magnitude of Some Heat-Transfer Coefficients, page 241

A.3-16 Thermal Conductivities, Densities, and Heat Capacities of Metals

Material	t (°C)	ρ $\left(\frac{kg}{m^3}\right)$	c_p $\left(\frac{kJ}{kg \cdot K}\right)$	k(W/m · K)		
Aluminum	20	2707	0.896	202 (0°C) 230 (300°C)	206 (100°C)	215 (200°C)
Brass (70–30)	20	8522	0.385	97 (0°C)	104 (100°C)	109 (200°C)
Cast iron	20	7593	0.465	55 (0°C)	52 (100°C)	48 (200°C)
Copper	20	8954	0.383	388 (0°C)	377 (100°C)	372 (200°C)
Lead	20	11 370	0.130	35 (0°C)	33 (100°C)	31 (200°C)
Steel 1%C	20	7801	0.473	45.3 (18°C) 43 (300°C)	45 (100°C)	45 (200°C)
308 stainless	20	7849	0.461	15.2 (100°C)	21.6 (500°C)	
304 stainless	0	7817	0.461	13.8 (0°C)	16.3 (100°C)	18.9 (300°C)
Tin	20	7304	0.227	62 (0°C)	59 (100°C)	57 (200°C)

Source: L. S. Marks, *Mechanical Engineers' Handbook*, 5th ed. New York: McGraw-Hill Book Company, 1951; E. R. G. Eckert and R. M. Drake, *Heat and Mass Transfer*, 2nd ed. New York: McGraw-Hill Book Company, 1959; R. H. Perry and C. H. Chilton, *Chemical Engineers' Handbook*, 5th ed. New York: McGraw-Hill Book Company, 1973; National Research Council, *International Critical Tables*. New York: McGraw-Hill Book Company, 1929.

The **Equation of Energy** in Cartesian, cylindrical, and spherical coordinates for Newtonian fluids of constant density, with source term S_e. Source could be electrical energy due to current flow, chemical energy, etc. Two cases are presented: the general case where thermal conductivity may be a function of temperature (vector flux $\underline{\tilde{q}} = \underline{q}/area$ appears in the equations); and the more usual case, where thermal conductivity is constant (on the reverse).

Spring 2020 Faith A. Morrison, Michigan Technological University

Microscopic energy balance, in terms of flux; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = -\nabla \cdot \underline{\tilde{q}} + S_e$$

Microscopic energy balance, in terms of flux; Cartesian coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = -\left(\frac{\partial \tilde{q}_x}{\partial x} + \frac{\partial \tilde{q}_y}{\partial y} + \frac{\partial \tilde{q}_z}{\partial z} \right) + S_e$$

Microscopic energy balance, in terms of flux; cylindrical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = -\left(\frac{1}{r} \frac{\partial (r \tilde{q}_r)}{\partial r} + \frac{1}{r} \frac{\partial \tilde{q}_\theta}{\partial \theta} + \frac{\partial \tilde{q}_z}{\partial z} \right) + S_e$$

Microscopic energy balance, in terms of flux; spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right) = -\left(\frac{1}{r^2} \frac{\partial (r^2 \tilde{q}_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\tilde{q}_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \tilde{q}_\phi}{\partial \phi} \right) + S_e$$

Fourier's law of heat conduction, Gibbs notation: $\underline{\tilde{q}} = \underline{q}/A = -k\nabla T$

Fourier's law of heat conduction, Cartesian coordinates:
(constant thermal conductivity k)

$$\begin{pmatrix} \tilde{q}_x \\ \tilde{q}_y \\ \tilde{q}_z \end{pmatrix}_{xyz} = \begin{pmatrix} q_x/A \\ q_y/A \\ q_z/A \end{pmatrix}_{xyz} = \begin{pmatrix} -k \frac{\partial T}{\partial x} \\ -k \frac{\partial T}{\partial y} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{xyz}$$

Fourier's law of heat conduction, cylindrical coordinates:
(constant thermal conductivity k)

$$\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} q_r/A \\ q_\theta/A \\ q_z/A \end{pmatrix}_{r\theta z} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{r\theta z}$$

Fourier's law of heat conduction, spherical coordinates:
(constant thermal conductivity k)

$$\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_\phi \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} q_r/A \\ q_\theta/A \\ q_\phi/A \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -\frac{k}{r \sin\theta} \frac{\partial T}{\partial \phi} \end{pmatrix}_{r\theta\phi}$$

The **Equation of Energy** for systems with **constant k**

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S_e$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\begin{aligned} \rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) \\ = k \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S_e \end{aligned}$$

Reference: F. A. Morrison, "Web Appendix to *An Introduction to Fluid Mechanics*," Cambridge University Press, New York, 2013 (*pages.mtu.edu/~fmorriso/cm310/IFMWebAppendixDMicroEBalanceMorrison.pdf*). This worksheet is on the web at *pages.mtu.edu/~fmorriso/cm310/energy.pdf*.

A.2-11 Heat-Transfer Properties of Liquid Water, SI Units

T (°C)	T (K)	ρ (kg/m³)	c_p (kJ/kg · K)	$\mu \times 10^3$ (Pa · s, or kg/m · s)	k (W/m · K)	N_{Pr}	$\beta \times 10^4$ (1/K)	$(g\beta\rho^2/\mu^2) \times 10^{-8}$ (1/K · m³)
0	273.2	999.6	4.229	1.786	0.5694	13.3	−0.630	
15.6	288.8	998.0	4.187	1.131	0.5884	8.07	1.44	10.93
26.7	299.9	996.4	4.183	0.860	0.6109	5.89	2.34	30.70
37.8	311.0	994.7	4.183	0.682	0.6283	4.51	3.24	68.0
65.6	338.8	981.9	4.187	0.432	0.6629	2.72	5.04	256.2
93.3	366.5	962.7	4.229	0.3066	0.6802	1.91	6.66	642
121.1	394.3	943.5	4.271	0.2381	0.6836	1.49	8.46	1300
148.9	422.1	917.9	4.312	0.1935	0.6836	1.22	10.08	2231
204.4	477.6	858.6	4.522	0.1384	0.6611	0.950	14.04	5308
260.0	533.2	784.9	4.982	0.1042	0.6040	0.859	19.8	11 030
315.6	588.8	679.2	6.322	0.0862	0.5071	1.07	31.5	19 260

A.2-11 Heat-Transfer Properties of Liquid Water, English Units

T (°F)	ρ $\left(\frac{lb_m}{ft^3}\right)$	c_p $\left(\frac{btu}{lb_m \cdot °F}\right)$	$\mu \times 10^3$ $\left(\frac{lb_m}{ft \cdot s}\right)$	k $\left(\frac{btu}{h \cdot ft \cdot °F}\right)$	N_{Pr}	$\beta \times 10^4$ (1/°R)	$(g\beta\rho^2/\mu^2) \times 10^{-6}$ (1/°R · ft³)
32	62.4	1.01	1.20	0.329	13.3	−0.350	
60	62.3	1.00	0.760	0.340	8.07	0.800	17.2
80	62.2	0.999	0.578	0.353	5.89	1.30	48.3
100	62.1	0.999	0.458	0.363	4.51	1.80	107
150	61.3	1.00	0.290	0.383	2.72	2.80	403
200	60.1	1.01	0.206	0.393	1.91	3.70	1010
250	58.9	1.02	0.160	0.395	1.49	4.70	2045
300	57.3	1.03	0.130	0.395	1.22	5.60	3510
400	53.6	1.08	0.0930	0.382	0.950	7.80	8350
500	49.0	1.19	0.0700	0.349	0.859	11.0	17 350
600	42.4	1.51	0.0579	0.293	1.07	17.5	30 300

Geankoplis, 4th edition

NOTE: Equate the label to the provided quantity in the supplied units. For example, for water at 0^oC:

$$\mu \times 10^3 = 1.786\ Pa\ s$$

$$\mu = 1.786 \times 10^{-3} Pa\ s$$

A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), SI Units

T (°C)	T (K)	ρ (kg/m^3)	c_p ($kJ/kg \cdot K$)	$\mu \times 10^5$ ($Pa \cdot s$, or $kg/m \cdot s$)	k ($W/m \cdot K$)	N_{Pr}	$\beta \times 10^3$ ($1/K$)	$g\beta\rho^2/\mu^2$ ($1/K \cdot m^3$)
−17.8	255.4	1.379	1.0048	1.62	0.02250	0.720	3.92	2.79×10^8
0	273.2	1.293	1.0048	1.72	0.02423	0.715	3.65	2.04×10^8
10.0	283.2	1.246	1.0048	1.78	0.02492	0.713	3.53	1.72×10^8
37.8	311.0	1.137	1.0048	1.90	0.02700	0.705	3.22	1.12×10^8
65.6	338.8	1.043	1.0090	2.03	0.02925	0.702	2.95	0.775×10^8
93.3	366.5	0.964	1.0090	2.15	0.03115	0.694	2.74	0.534×10^8
121.1	394.3	0.895	1.0132	2.27	0.03323	0.692	2.54	0.386×10^8
148.9	422.1	0.838	1.0174	2.37	0.03531	0.689	2.38	0.289×10^8
176.7	449.9	0.785	1.0216	2.50	0.03721	0.687	2.21	0.214×10^8
204.4	477.6	0.740	1.0258	2.60	0.03894	0.686	2.09	0.168×10^8
232.2	505.4	0.700	1.0300	2.71	0.04084	0.684	1.98	0.130×10^8
260.0	533.2	0.662	1.0341	2.80	0.04258	0.680	1.87	0.104×10^8

A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), English Units

T (°F)	ρ $\left(\frac{lb_m}{ft^3}\right)$	c_p $\left(\frac{btu}{lb_m \cdot °F}\right)$	μ (centipoise)	k $\left(\frac{btu}{h \cdot ft \cdot °F}\right)$	N_{Pr}	$\beta \times 10^3$ ($1/°R$)	$g\beta\rho^2/\mu^2$ ($1/°R \cdot ft^3$)
0	0.0861	0.240	0.0162	0.0130	0.720	2.18	4.39×10^6
32	0.0807	0.240	0.0172	0.0140	0.715	2.03	3.21×10^6
50	0.0778	0.240	0.0178	0.0144	0.713	1.96	2.70×10^6
100	0.0710	0.240	0.0190	0.0156	0.705	1.79	1.76×10^6
150	0.0651	0.241	0.0203	0.0169	0.702	1.64	1.22×10^6
200	0.0602	0.241	0.0215	0.0180	0.694	1.52	0.840×10^6
250	0.0559	0.242	0.0227	0.0192	0.692	1.41	0.607×10^6
300	0.0523	0.243	0.0237	0.0204	0.689	1.32	0.454×10^6
350	0.0490	0.244	0.0250	0.0215	0.687	1.23	0.336×10^6
400	0.0462	0.245	0.0260	0.0225	0.686	1.16	0.264×10^6
450	0.0437	0.246	0.0271	0.0236	0.674	1.10	0.204×10^6
500	0.0413	0.247	0.0280	0.0246	0.680	1.04	0.163×10^6

Source: National Bureau of Standards, *Circular* 461C, 1947; 564, 1955; NBS-NACA, *Tables of Thermal Properties of Gases*, 1949; F. G. Keyes, *Trans. A.S.M.E.*, 73, 590, 597 (1951); 74, 1303 (1952); D. D. Wagman, *Selected Values of Chemical Thermodynamic Properties*. Washington, D.C.: National Bureau of Standards, 1953.

Geankoplis, 4th edition

NOTE: Equate the label to the provided quantity in the supplied units. For example, for air at $0°C$:

$$\mu \times 10^5 = 1.72\ Pa\ s$$

$$\mu = 1.72 \times 10^{-5} Pa\ s$$

Heat Transfer Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
Professor Faith A. Morrison
1 December 2020

I. Forced Convection Through Pipes

In forced convection, we determined from dimensional analysis that the Nusselt number is a function of at most Re, Pr, L/D, and viscosity ratio.

Prandtl number (fluid properties)
$$\mathrm{Pr} \equiv \frac{\hat{C}_p \mu}{k} \tag{1}$$

In pipe flow with heat transfer taking place, the fluid enters at bulk fluid temperature T_{bi} and exits at T_{bo}. T_w is the temperature of the wall. For Nu data correlations in forced convection through pipes, all fluid material properties except $\mu_w = \mu(T_w)$ are evaluated at the mean bulk temperature. The mean bulk temperature is given by

Mean bulk temperature
$$\bar{T}_b \equiv \frac{T_{bi} + T_{bo}}{2} \tag{2}$$

A. Laminar Flow in Pipes

Sieder and Tate's correlation (Geankoplis, p260) for laminar flow is

Laminar flow
$$\mathrm{Nu}_a = \frac{h_a D}{k} = 1.86 \left(\mathrm{RePr}\frac{D}{L}\right)^{\frac{1}{3}} \left(\frac{\mu_b}{\mu_w}\right)^{0.14} \tag{3}$$

$$q = h_a A \Delta T_a \tag{4}$$

Arithmetic mean driving force
$$\Delta T_a = \frac{(T_w - T_{bi}) + (T_w - T_{bo})}{2} \tag{5}$$

B. Turbulent Flow in Pipes

Sieder and Tate's correlation (Geankoplis, p261) for turbulent flow is

Turbulent flow
$$\mathrm{Nu}_{lm} = \frac{h_{lm} D}{k} = 0.027 \mathrm{Re}^{0.8} \mathrm{Pr}^{\frac{1}{3}} \left(\frac{\mu_b}{\mu_w}\right)^{0.14} \tag{6}$$

$$q = h_{lm} A \Delta T_{lm} \tag{7}$$

Log mean driving force
$$\Delta T_{lm} = \frac{\Delta T_{w-bi} - \Delta T_{w-bo}}{ln\left(\frac{\Delta T_{w-bi}}{\Delta T_{w-bo}}\right)} \tag{8}$$

II. Forced Convection Around the Outside of a Cylinder

In heat transfer taking place between a fluid at bulk temperature T_b flowing perpendicular to a cylinder with wall temperature T_w, the material properties are evaluated at the film temperature.

Film temperature
$$T_f \equiv \frac{T_w + T_b}{2} \tag{9}$$

The data correlation for Nusselt number in this case is

Outside Cylinder
$$\mathrm{Nu} = \frac{hD}{k} = C\mathrm{Re}^m \mathrm{Pr}^{\frac{1}{3}} \tag{10}$$

Wall-bulk driving force
$$q = hA(T_w - T_b) \tag{11}$$

The values of C and m depend on the Reynolds number (Geankoplis, Table 4.6-1, p272). These values are valid for $\mathrm{Pr} > 0.6$.

Re	m	C
$1 - 4$	0.330	0.989
$4 - 40$	0.385	0.911
$40 - 4{,}000$	0.466	0.683
$4{,}000 - 4 \times 10^4$	0.618	0.193
$4 \times 10^4 - 2.5 \times 10^5$	0.805	0.0266

III. Natural Convection from Various Geometries

Natural convection heat transfer coefficients from various surfaces have been found by dimensional analysis and experimentally to correlate as follows:

Natural convection (various geometries) $$\mathrm{Nu} = \frac{hL}{k} = a(\mathrm{Gr\,Pr})^m \quad (12)$$

Grashof number $$\mathrm{Gr} \equiv \frac{L^3\rho^2 g\beta\Delta T}{\mu^2} \quad (13)$$

The values for a and m depend on the geometry; values may be found in Geankoplis in Table 4.7-1 (p278, shown below). Table 4.7-2 (p280, next pages) provides simplified versions of the correlations specialized to common fluids (air, water, organic liquids).

TABLE 4.7-1. *Constants for Use with Eq. (4.7-4) for Natural Convection*

Physical Geometry	$N_{Gr}N_{Pr}$	a	m	*Ref.*
Vertical planes and cylinders [vertical height $L < 1$ m (3 ft)]				
	$<10^4$	1.36	$\frac{1}{5}$	(P3)
	10^4–10^9	0.59	$\frac{1}{4}$	(M1)
	$>10^9$	0.13	$\frac{1}{3}$	(M1)
Horizontal cylinders [diameter D used for L and $D < 0.20$ m (0.66 ft)]				
	$<10^{-5}$	0.49	0	(P3)
	10^{-5}–10^{-3}	0.71	$\frac{1}{25}$	(P3)
	10^{-3}–1	1.09	$\frac{1}{10}$	(P3)
	1–10^4	1.09	$\frac{1}{5}$	(P3)
	10^4–10^9	0.53	$\frac{1}{4}$	(M1)
	$>10^9$	0.13	$\frac{1}{3}$	(P3)
Horizontal plates				
Upper surface of heated plates or lower surface of cooled plates	10^5–2×10^7	0.54	$\frac{1}{4}$	(M1)
	2×10^7–3×10^{10}	0.14	$\frac{1}{3}$	(M1)
Lower surface of heated plates or upper surface of cooled plates	10^5–10^{11}	0.58	$\frac{1}{5}$	(F1)

TABLE 4.7-2. *Simplified Equations for Natural Convection from Various Surfaces*

Physical Geometry	$N_{Gr}N_{Pr}$	*Equation*: $h = btu/h \cdot ft^2 \cdot °F$, $L = ft$, $\Delta T = °F$, $D = ft$	*Equation*: $h = W/m^2 \cdot K$, $L = m$, $\Delta T = K$, $D = m$	*Ref.*
Air at 101.32 kPa (1 atm) abs pressure				
Vertical planes and cylinders	10^4–10^9	$h = 0.28(\Delta T/L)^{1/4}$	$h = 1.37(\Delta T/L)^{1/4}$	(P1)
	$>10^9$	$h = 0.18(\Delta T)^{1/3}$	$h = 1.24\,\Delta T^{1/3}$	(P1)
Horizontal cylinders	10^3–10^9	$h = 0.27(\Delta T/D)^{1/4}$	$h = 1.32(\Delta T/D)^{1/4}$	(M1)
	$>10^9$	$h = 0.18(\Delta T)^{1/3}$	$h = 1.24\,\Delta T^{1/3}$	(M1)
Horizontal plates				
Heated plate facing upward or cooled plate facing downward	10^5–2×10^7	$h = 0.27(\Delta T/L)^{1/4}$	$h = 1.32(\Delta T/L)^{1/4}$	(M1)
	2×10^7–3×10^{10}	$h = 0.22(\Delta T)^{1/3}$	$h = 1.52\,\Delta T^{1/3}$	(M1)
Heated plate facing downward or cooled plate facing upward	3×10^5–3×10^{10}	$h = 0.12(\Delta T/L)^{1/4}$	$h = 0.59(\Delta T/L)^{1/4}$	(M1)
Water at 70°F (294 K)				
Vertical planes and cylinders	10^4–10^9	$h = 26(\Delta T/L)^{1/4}$	$h = 127(\Delta T/L)^{1/4}$	(P1)
Organic liquids at 70°F (294 K)				
Vertical planes and cylinders	10^4–10^9	$h = 12(\Delta T/L)^{1/4}$	$h = 59(\Delta T/L)^{1/4}$	(P1)

Reference: C. J. Geankoplis, Transport Processes and Separation Process Principles, 4th Edition, Prentice Hall, 2003.

Mechanism	$h, \frac{BTU}{hr\ ft^{2\,0}F}$	$h, \frac{W}{m^2K}$
Condensing steam	1000-5000	5700-28,000
Condensing organics	200-500	1100-2800
Boiling liquids	300-5000	1700-28,000
Moving water	50-3000	280-17,000
Moving hydrocarbons	10-300	55-1700
Still air	0.5-4	2.8-23
Moving air	2-10	11.3-55

Reference: C. J. Geankoplis, Magnitude of Some Heat-Transfer Coefficients, page 241

Table A.3-16 Geankoplis, 2003

A.3-16 Thermal Conductivities, Densities, and Heat Capacities of Metals

Material	t (°C)	ρ $\left(\frac{kg}{m^3}\right)$	c_p $\left(\frac{kJ}{kg\cdot K}\right)$	$k(W/m\cdot K)$		
Aluminum	20	2707	0.896	202 (0°C) 230 (300°C)	206 (100°C)	215 (200°C)
Brass (70–30)	20	8522	0.385	97 (0°C)	104 (100°C)	109 (200°C)
Cast iron	20	7593	0.465	55 (0°C)	52 (100°C)	48 (200°C)
Copper	20	8954	0.383	388 (0°C)	377 (100°C)	372 (200°C)
Lead	20	11 370	0.130	35 (0°C)	33 (100°C)	31 (200°C)
Steel 1%C	20	7801	0.473	45.3 (18°C) 43 (300°C)	45 (100°C)	45 (200°C)
308 stainless	20	7849	0.461	15.2 (100°C)	21.6 (500°C)	
304 stainless	0	7817	0.461	13.8 (0°C)	16.3 (100°C)	18.9 (300°C)
Tin	20	7304	0.227	62 (0°C)	59 (100°C)	57 (200°C)

Source: L. S. Marks, *Mechanical Engineers' Handbook*, 5th ed. New York: McGraw-Hill Book Company, 1951; E. R. G. Eckert and R. M. Drake, *Heat and Mass Transfer*, 2nd ed. New York: McGraw-Hill Book Company, 1959; R. H. Perry and C. H. Chilton, *Chemical Engineers' Handbook*, 5th ed. New York: McGraw-Hill Book Company, 1973; National Research Council, *International Critical Tables*. New York: McGraw-Hill Book Company, 1929.