Calculating Fluid Forces on Surfaces in Comsol 5.1

Professor Faith Morrison
Michigan Technological University
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To calculate the fluid force on a surface, $\underline{F}$, from a solution to the microscopic momentum balance (the Navier-Stokes equation), we use the following expression (Morrison, 2013):

$$\underline{F} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]\Big|_{surface} dS \tag{1}$$

where S is the area of the macroscopic surface, $\underline{\underline{\tilde{\Pi}}}$ is the total stress tensor, dS is a differential surface element on the surface, and $\hat{n}$ is the unit normal to dS. The tensor $\underline{\underline{\tilde{\tau}}}$ is the extra stress tensor, also called the viscous stress tensor, which for a Newtonian fluid it is related to the velocity field $\underline{v}(x, y, z)$ as follows:

$$\underline{\underline{\tilde{\tau}}} = \mu(\nabla\underline{v} + (\nabla\underline{v})^T) \tag{2}$$

$$\underline{\underline{\tilde{\Pi}}} = \underline{\underline{\tilde{\tau}}} - p\underline{\underline{I}} \tag{3}$$

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{xyz} \tag{4}$$

$$\underline{\underline{\tilde{\tau}}} = \begin{pmatrix} 2\mu\frac{dv_x}{dx} & \mu\left(\frac{dv_x}{dy} + \frac{dv_y}{dx}\right) & \mu\left(\frac{dv_x}{dz} + \frac{dv_z}{dx}\right) \\ \mu\left(\frac{dv_x}{dy} + \frac{dv_y}{dx}\right) & 2\mu\frac{dv_y}{dy} & \mu\left(\frac{dy}{dz} + \frac{dv_z}{dy}\right) \\ \mu\left(\frac{dv_x}{dz} + \frac{dv_z}{dx}\right) & \mu\left(\frac{dy}{dz} + \frac{dv_z}{dy}\right) & 2\mu\frac{dv_z}{dz} \end{pmatrix}_{xyz} \tag{5}$$

In Comsol, the software calculates the velocity field at every point on a mesh that the user chooses. To calculate the total force on a surface $\underline{F}$ it must calculate the integral in Equation 1 from the numerical solution for the velocity $\underline{v}(x, y, z)$ (stored in matricies for every location) and the relationships in Equations 2-5.

The kernel of the integral in equation 1 is a vector with units of stress (Pa). In Comsol, the coefficients of this stress vector are given the following names:

Total stress vector on a surface at a location with unit normal $\hat{n}$

$$\underline{T} \equiv \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]\Big|_{surface} = \begin{pmatrix} \text{spf.T\_stressx} \\ \text{spf.T\_stressy} \\ \text{spf.T\_stressz} \end{pmatrix}_{xyz} \tag{6}$$

A related stress vector is the analogous expression involving the tensor $\underline{\underline{\tilde{\tau}}}$. In Comsol, this is given by

Viscous stress vector on a surface at a location with unit normal $\hat{n}$

$$\underline{K} \equiv \left[\hat{n} \cdot \underline{\underline{\tilde{\tau}}}\right]\Big|_{surface} = \begin{pmatrix} \text{spf.K\_stressx} \\ \text{spf.K\_stressy} \\ \text{spf.K\_stressz} \end{pmatrix}_{xyz} \tag{7}$$

From Equation 1, we see that to calculate the fluid force on a surface Comsol needs to integrate the stress vector $\underline{T}$ over the surface. We can write this as

$$\underline{F} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]\Big|_{surface} dS = \iint_S \underline{T}\, dS \tag{8}$$

$$\underline{F} = \iint_S \begin{pmatrix} \text{spf.T\_stressx} \\ \text{spf.T\_stressy} \\ \text{spf.T\_stressz} \end{pmatrix}_{xyz} dS \tag{9}$$

Note that this is the integration of stress=force/area over an area and thus yields a force (N).

In a 2D calculation, we consider only a slice of the flow. For a 2D rectangular geometry, for example, we may only be considering flow in the y direction as a function of the x-direction, with no variations taking place in the z-direction. For such a flow, the calculation for $\underline{F}$ on a side wall (in a yz-plane, length L, width W) becomes

$$\underline{F} = \int_0^W \int_0^L \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]\Big|_{surface} dydz \tag{10}$$

$$\underline{F} = \int_0^W \int_0^L \begin{pmatrix} \text{spf.T\_stressx} \\ \text{spf.T\_stressy} \\ \text{spf.T\_stressz} \end{pmatrix}_{xyz} dydz \tag{11}$$

Since there is no variation of velocity in the z-direction, we may carry out the z integration, obtaining

$$\underline{F} = W \int_0^L \begin{pmatrix} \text{spf.T\_stressx} \\ \text{spf.T\_stressy} \\ \text{spf.T\_stressz} \end{pmatrix}_{xyz} dy \tag{12}$$

$$\frac{\underline{F}}{W} = \int_0^L \begin{pmatrix} \text{spf.T\_stressx} \\ \text{spf.T\_stressy} \\ \text{spf.T\_stressz} \end{pmatrix}_{xyz} dy \tag{13}$$

Note that Equation 13 is the integration of stress=force/area over a single direction (y) and thus yields a force/length (N/m). In Comsol 5.1 the calculation in Equation 13 is accomplished through steps related to these choices: Derived Values-Line Integration-select the surface, select the appropriate stress vector component, spf.T_stressx, spf.T_stressy, or spf.T_stressz (one at a time.)

Reference:

Morrison, Faith A, *An Introduction to Fluid Mechanics* (Cambridge University Press, New York, 2013).