3.3 Summary

In this chapter, we take the first steps toward developing a problem-solving method for two types of flow problems: microscopic and macroscopic. We now summarize our progress.

The continuum model is a way of viewing fluids using a set of continuous functions to keep track of fluid behavior, ignoring molecular details. The continuous functions of fluid mechanics include the density field, the velocity field, and the molecular-stress field, which is discussed in Chapter 4. Calculus is the mathematics of continuous functions and we use it extensively to make our calculations of fluid motion and fluid forces.

Fluid motion is governed by mass, momentum, and energy balances. We choose to use balances on control volumes instead of on individual bodies. The control-volume method is more convenient to use in fluid mechanics because fluids are not individual rigid bodies like those with which we deal in introductory physics and mechanics courses. The control-volume method is well suited for use with the continuum picture, as shown in the final two examples in this chapter. We continue study of these two problems in Chapters 4 and 5 and consider more problems of this type in Chapters 7–10.

The appropriate momentum balance to use with a control volume is given by the Reynolds transport theorem:

Reynolds transport theorem (momentum balance on CV)

$$\frac{d\underline{\mathbf{P}}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v})\, \rho \underline{v}\, dS = \sum_{\substack{\text{on}\\ \text{CV}}} \underline{f} \qquad (3.188)$$

Recall that $\hat{n}$ is the outwardly pointing normal to the CV enclosing surface CS; thus, the integral in Equation 3.188 is net outflow of momentum from the CV.

To apply the Reynolds transport theorem to a problem, we must be able to identify the forces that are acting on the CV, including molecular forces. In this chapter, we discuss one force—gravity—that acts on a CV. Chapter 4 introduces molecular stress, the source of a second significant force that acts on a CV. In Chapter 5, we discuss the link between molecular stress and fluid motion. When these topics have been covered, we can complete our flow calculations on the inclined plane and the 90-degree bend, and we will be ready to tackle a wide variety of problems in fluid mechanics.

3.4 Problems

1. What is a control volume? Why does the field of fluid mechanics introduce this concept?
2. What is a fluid particle? How big is a fluid particle?
3. How is the concept of a continuum different from your understanding of matter from chemistry studies?
4. What is meant by the term *velocity field*? What other "fields" are there in fluid mechanics and physics?

Table 3.2. Data of $y(t)$ for Problem 11

t (s)	y (m/s)
3.6	3.9
4.0	12.1
4.4	22.7
4.8	33.0
5.2	41.9
5.6	49.7
6.0	55.6
6.4	61.0
6.8	65.5
7.4	71.0
8.2	76.7
9.2	82.0
10.4	86.2
12.0	90.2
13.4	92.7
15.4	94.6
17.4	95.9
19.4	97.1
21.4	98.0
22.8	98.4
24.4	98.5
26.0	98.7
27.6	99.0
29.4	100.0
31.2	100.2
33.6	100.7

5. What are the principal forces that cause flow?
6. What is Newton's second law $\sum \underline{f} = m\underline{a}$ when written on a control volume V with bounding surfaces CS?
7. We derived the Reynolds transport theorem for the momentum balance. What is it for the mass balance?
8. Why are we unable to use the momentum balance $\sum \underline{f} = m\underline{a}$ (i.e., Newton's second law) directly in fluid-flow calculations?
9. What is the difference between the rate of change of momentum terms $\frac{d(m\underline{v})}{dt}$ and $\frac{d\mathbf{P}}{dt}$ in Newton's second law (Equation 3.52) and the Reynolds transport theorem (Equation 3.135)?
10. In Equation 3.126 in the development of the convective term of the momentum balance, an indeterminate vector product ($\underline{v}\ \underline{v}$) appears. How did that expression come to include a dyadic product? What is the meaning of the tensor $\rho\underline{v}\ \underline{v}$?
11. For the data given in Table 3.2 (i.e., arbitrary time-dependent quantity y), find a function $y(t)$ that fits the data well. What is your estimate of y (7.0)?
12. For the experimental data given in Table 3.3 (i.e., viscosity of aqueous sugar solutions as a function of concentration), find a function $\mu(c)$ that fits the data well. What is your estimate of μ (28.2 wt%) and μ (50.0 wt%)?

Table 3.3. Experimental data of viscosity as a function of concentration $\mu(c)$ of aqueous sugar solutions for Problem 12

c (wt% sugar)	μ (cp)
10	0.62
10	0.87
10	0.88
10	0.89
20	1.0
20	1.2
20	1.2
20	1.2
20	1.2
20	1.3
30	2.0
30	2.1
30	2.1
30	2.3
30	3.0
40	3.8
40	4.3
40	4.3
40	4.4
40	4.6
45	5.2
45	5.3
45	5.3
50	8.4
50	9.3
50	9.5
50	9.7
50	14
60	28
60	30
60	30
60	32
65	63
65	64
65	65
65	69

13. For the experimental data given in Table 3.4 (i.e., pumping head as a function of volumetric flow rate [102]), find a function $H_{pump}(Q)$ that fits the data well. How much head does the pump develop at 2.2 gpm?
14. A uniform flow $\underline{v} = U\hat{e}_z$ of an incompressible fluid of density ρ passes through a volume that is in the shape of a half sphere of radius R. The outwardly pointing unit normal of the flat surface of the half sphere is $\hat{n} = -\hat{e}_z$. What is the mass flow rate through the hemispherical surface of this volume? Show that you can obtain the correct answer by integrating the formal expression for Q (Equation 3.87).
15. What is the flow of momentum through the hemispherical surface described in Problem 14?

Table 3.4. Experimental data of pumping head as a function of volumetric flow rate $H_{pump}(Q)$ for a laboratory pump [102] (Problem 13)

Q gpm	Head ft
0.88	72.5
1.00	68.1
1.38	70.5
1.87	67.2
1.99	70.4
2.37	63.6
2.86	58.9
3.23	57.3
3.36	52.7
3.85	46.2

16. A uniform flow $\underline{v} = U\hat{e}_x$ of an incompressible fluid of density ρ passes through a volume that is in the shape of a block (i.e., rectangular parallelepiped). The sides of the block are lengths $a < b < c$. The unit normal to the cb surface is $\hat{n} = (\hat{e}_x - \hat{e}_y)/\sqrt{2}$. What is the mass flow rate through the cb surface? What is the momentum flow rate through the cb surface?
17. For the volume described in Problem 16, what are the unit normals to the other two surfaces?
18. For the volume described in Problem 16, what is the mass flow rate through the ac surface? What is the momentum flow rate through the ac surface?
19. For the function $f(x)$ given here, what is the average value $\langle f \rangle$ of the function between $x = 0$ and $x = 2$?

$$f(x) = 2x^2 + 3$$

20. For the velocity-profile function $v_y(x)$ given here (equation uses Cartesian coordinates xyz, $\underline{v} = v_y\hat{e}_y$), what is the average value $\langle v_y \rangle$ of the function between $x = 0$ and $x = 2$? The units of velocity are m/s and the units of x are m.

$$v_y(x) = 3\left(\frac{x}{6}\right)^2 + 1.5$$

21. The y-component of a velocity field in flow through a slit (equation uses Cartesian coordinates) is given here. What is the average value of the velocity? $2H$ is the gap between the plates. At what location is the velocity a maximum? The units of velocity and A are m/s and the units of x and H are m.

$$\underline{v} = \begin{pmatrix} 0 \\ v_y(x) \\ 0 \end{pmatrix}_{xyz}$$

$$v_y(x) = A\left(1 - \frac{(x - H)^2}{H^2}\right)$$

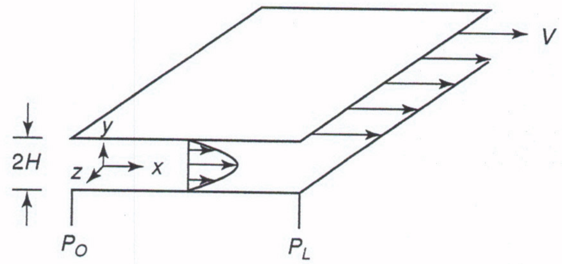

Figure 3.37 Pressure-driven flow (i.e., Poiseuille flow) through a slit with a superimposed drag flow due to the motion of the top plate (Problem 24).

22. The z-component of a velocity field in flow through a tube (given in cylindrical coordinates) is shown here. What is the average value of the velocity? R is the radius of the tube. The units of velocity and A are m/s and the units of r and R are m.

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z(r) \end{pmatrix}_{r\theta z}$$

$$v_z = A\left(1 - \frac{r^2}{R^2}\right)$$

23. For the velocity-profile function $v_z(r)$ given here, what is the average value $\langle v_z \rangle$ of the function between $r = 5$ and $r = 10$? The function is written in cylindrical coordinates. The units of velocity are m/s and the units of r are m.

$$v_z(r) = 8 \ln\left(\frac{r}{3}\right)$$

24. The x-component of a velocity field is given here (expressed in Cartesian coordinates). This velocity profile results from pressure-driven flow through a slit with the top wall moving at velocity V (Figure 3.37). What is the average value of the velocity? $2H$ is the gap between the plates, a pressure gradient $\Delta P/L$ is imposed, and the fluid viscosity is μ. At what location is the velocity a maximum?

$$\underline{v} = \begin{pmatrix} v_x(y) \\ 0 \\ 0 \end{pmatrix}_{xyz}$$

$$v_x(y) = \left(\frac{H^2(\Delta P)}{2\mu L}\right)\left(1 - \frac{y^2}{H^2}\right) + \frac{V}{2}\left(1 + \frac{y}{H}\right)$$

25. What is the wetted surface area of water flowing in a tube? Show that you can obtain the answer by performing an integration in cylindrical coordinates.
26. What is the wetted surface area of a sphere dropping in a fluid? Show that you can obtain the answer by integrating an appropriate quantity.

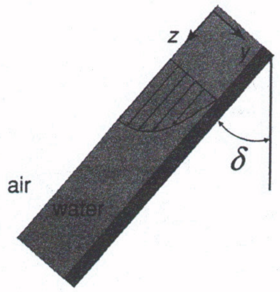

Figure 3.38 Flow coordinate system for Problem 31.

27. What is the wetted surface area of an open, semicircular channel (i.e., half pipe) of length L and pipe radius R, in which the fluid height in the center is h. Show that you can obtain the answer by integrating an appropriate quantity.
28. For a pipe that is only 80 percent full (i.e., occupied volume = 80 percent of the total pipe volume), what is the wetted surface area? The pipe is of length L and radius R.
29. For the two vectors given here, what is $|\underline{w}|$? What is $|\underline{v}|$? What is $(\underline{w} \cdot \underline{v})$? What is the angle between the two vectors?

$$\underline{w} = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}_{123} \qquad \underline{v} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}_{123}$$

30. For the two vectors given here, what is $|\underline{w}|$? What is $|\underline{v}|$? What is $(\underline{w} \cdot \underline{v})$? What is the angle between the two vectors? Note that the two vectors are not written relative to the same coordinate system.

$$\underline{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{r\theta z}\Bigg|_{r=1,\theta=\pi,\phi=0} \qquad \underline{v} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}_{123}$$

31. For the Cartesian coordinate system shown in Figure 3.38, what is a unit vector in the direction of gravity? What is the component of gravity in the flow direction?
32. For the cylindrical coordinate system shown in Figure 3.39 for the axial flow in a wire-coating operation, what is a unit vector in the direction of gravity? What is the component of gravity in the flow direction?
33. For the horizontal flow around a sphere in a wind tunnel, the top view of the geometry is shown in Figure 3.40. Relative to the spherical coordinate system shown, what is a unit vector in the direction of gravity? What is the component of gravity in the flow direction?
34. For a particular problem, the control volume is chosen to be a rectangular parallelepiped of dimensions length L, width W, and height H. What is the total surface area of the control volume? What is the volume of the control volume? Choose a coordinate system and write formal surface integrals over the surfaces and verify your answer for total surface area. Write a formal volume integral and verify your answer for volume.

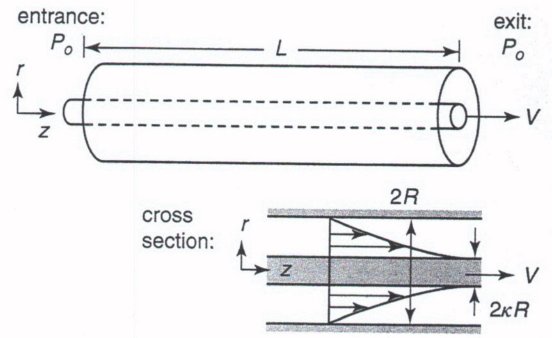

Figure 3.39 Axial annular flow that occurs in wire coating (Problem 32).

35. For a particular problem, the control volume is chosen to be a right-circular cylinder of radius R and height H. What is the total surface area of the control volume? What is the volume of the control volume? Choose a coordinate system and write formal surface integrals over the surfaces and verify your answer for total surface area. Write a formal volume integral and verify your answer for volume.
36. For a particular problem, the control volume is chosen to be a cone of height H and widest radius R. What is the total surface area of the control volume? What is the volume of the control volume? Choose a coordinate system and write formal surface integrals over the surfaces and verify your answer for total surface area. Write a formal volume integral and verify your answer for volume.
37. For a particular flow problem, the control volume is chosen to be a rectangular parallelepiped with dimensions of length L, width W, and height H. The Cartesian coordinate system chosen is located at one corner of the control volume ($0 \leq x \leq L$, $0 \leq y \leq W$, $0 \leq z \leq H$). For each enclosing control

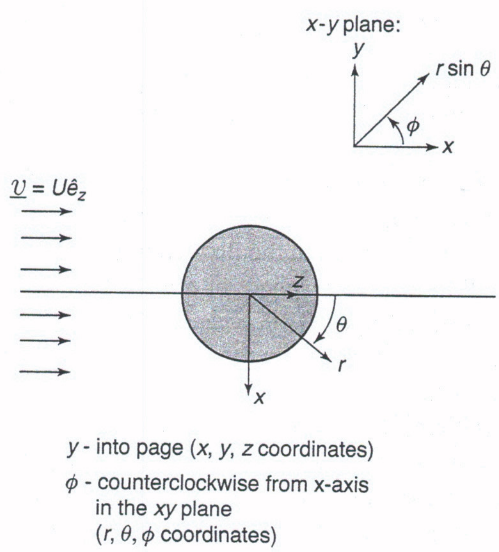

Figure 3.40 Flow around a sphere in a wind tunnel (Problem 33).

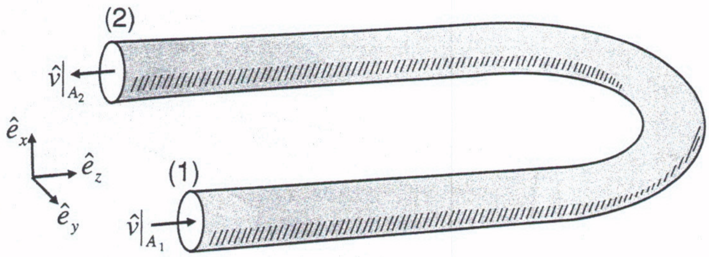

Figure 3.41 When fluid flows in a U-shaped tube, the momentum changes direction and forces are required to restrain the tube (Problems 41 and 44).

surface of this control volume, what are the outwardly pointing unit normal vectors $\hat{n}$ for each control surface? For a uniform flow $\underline{v} = U_o(\hat{e}_x + \hat{e}_y)$ through the control volume, what is the mass flow rate through each control surface? The fluid has constant density ρ and U_0 is constant.

38. For a particular flow problem, the control volume is chosen to be a vertical-right circular cylinder of radius R and height H. Choose a cylindrical coordinate system for flow down the cylindrical axis of this control volume. For each enclosing control surface of the control volume, write the unit vectors that are normal to each control surface. For a uniform flow $\underline{v} = U\hat{e}_z$ through the control volume (U is constant), what is the flow rate through each control surface? The fluid has variable density ρ. For a flow $\underline{v} = \left(U\frac{1}{r}\right)\hat{e}_r$ through the control volume, what is the mass flow rate through each control surface?
39. For a particular flow problem, the control volume is chosen to be a truncated cone of height H, bottom widest radius R_1, and top smaller radius R_2. The cone is truncated a distance l from the tip and the cone angle is $\theta = \alpha$, where θ is the coordinate variable for a spherical coordinate system with origin at the core tip. For each enclosing control surface, write the unit vectors that are normal to each control surface. For a uniform flow $\underline{v} = -U\hat{e}_z$ down the axis of the control volume, what is the mass flow rate through each control surface? The fluid has constant density ρ and the flow first passes through the bottom of the control volume.
40. An incompressible fluid (i.e., density is constant) enters a rectangular duct flowing at a steady flow rate of Q gpm. The width of the duct is W, the height of the duct is H, and the length of the duct is L. What is the average velocity of fluid entering the duct in terms of these variables? What is the average velocity of fluid exiting the duct?
41. An incompressible fluid (i.e., density is constant) enters a U-shaped conduit flowing at a steady flow rate of Q gpm (Figure 3.41). The conduit has a circular cross section all along its length and the radius of the conduit is R. What is the average velocity of fluid entering the conduit in terms of these variables? What is the average velocity of fluid exiting the conduit?
42. An incompressible fluid enters a converging bend flowing at a steady flow rate of Q gpm (Figure 3.42). The bend makes a 20-degree turn and has a circular cross section all along its length. At the inlet to the bend, the radius of the conduit is R_1; at the exit, the radius is a smaller value, R_2. What is

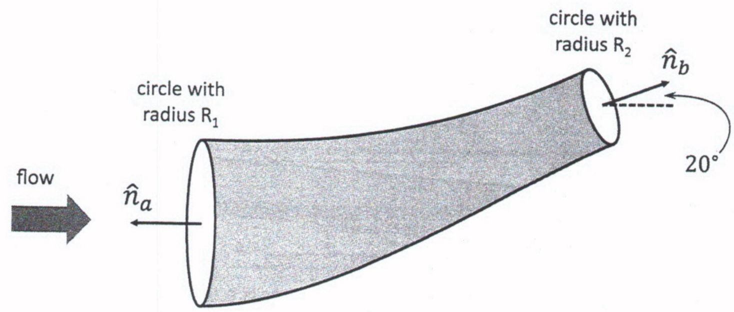

Figure 3.42 Schematic of a converging fitting (Problems 42 and 47).

the average velocity of fluid entering the conduit in terms of these variables? What is the average velocity of fluid exiting the conduit?

43. An incompressible fluid enters a horizontal, diverging conduit flowing at a steady flow rate of Q gpm. The conduit has a circular cross section all along its length. At the inlet, the radius of the conduit is R_1; at the exit, the radius is a larger value, R_2. What is the average velocity of fluid entering the conduit in terms of these variables? What is the average velocity of fluid exiting the conduit?
44. In this chapter, we introduced the Reynolds transport theorem:

$$\text{Reynolds transport theorem (momentum balance on CV)} \quad \sum_{\substack{\text{on}\\ \text{CV}}} \underline{f} = \frac{d\mathbf{\underline{P}}}{dt} + \iint_S (\hat{n} \cdot \underline{v})\, \rho \underline{v}\, dS$$

The convective term is the integral in the Reynolds transport theorem, and this term accounts for the net loss of momentum from the control volume through its bounding surfaces. Consider two cases of flow with an average inlet velocity of $\langle v \rangle$: (a) steady flow through a straight tube of radius R, and (b) steady flow through a U-shaped tube of radius R (see Figure 3.41). For Case (a), the convective term is zero; for Case (b), the convective term is not zero. Perform each calculation and explain the results.
45. In Equation 3.181 for the problem of flow in a right-angle bend, the convective term of the macroscopic momentum balance is not equal to zero, even though an equal magnitude of momentum enters and exits the control volume. Explain why this is so.
46. Evaluate the convective term of the Reynolds transport theorem for the 162-degree bend-reducing fitting shown in Figure 3.43. The flow is into the wider cross section.
47. Evaluate the convective term of the Reynolds transport theorem for the 20-degree bend-reducing fitting shown in Figure 3.42.
48. Set up the problem of steady flow of a Newtonian fluid down an inclined plane using a Cartesian coordinate system in which gravity is in the $(-z)$-direction.
49. Set up the problem of steady flow of a Newtonian fluid through a right-angle bend using a cylindrical coordinate system with the z-direction as the inlet flow direction. What is the velocity vector like at the exit for this

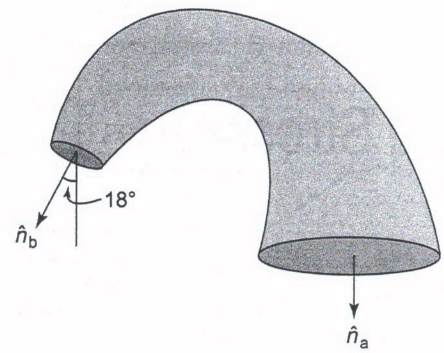

Figure 3.43 Schematic of a reducing fitting (Problem 46).

chosen coordinate system? What is the gravity vector? Comment on your observations.

50. The definition of a derivative is given in Chapter 1 (see Equation 1.138):

$$\frac{df}{dx} \equiv \lim_{\Delta x \longrightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

What is the derivative (df/dx) of $f(x) = x^2$? Formally verify your answer by plugging in $f(x)$ and $f(x + \Delta x)$ into the definition and carrying out the limit.