## 6.4 Problems

1. Compare and contrast the solution to the flow down an incline plane pursued in Chapters 3–5 with the solution in Example 6.2.
2. What is the difference between solving for pressure with the mechanical energy balance and solving for pressure with the Navier-Stokes equation?
3. In the derivation of the continuity equation, we omit details of some vector/tensor manipulations. Using matrix notation in a Cartesian coordinate system or using Einstein notation, show that Equation 6.41 may be simplified to give Equation 6.43.
4. In the derivation of the Cauchy momentum equation, we omit details of some vector/tensor manipulations. Using matrix notation in a Cartesian coordinate system or using Einstein notation, show that Equation 6.62 may be simplified to give Equation 6.63.
5. In the derivation of the Navier-Stokes equation, we omit details of some vector/tensor manipulations. Using matrix notation in a Cartesian coordinate system or using Einstein notation, show that Equation 6.68 may be simplified to give Equation 6.70.
6. In the calculation of the total flow rate down an inclined plane, integrate Equation 5.193 to obtain the final result.
7. Show that the results for creeping flow around a sphere (see Equation 5.101) satisfy the continuity equation for incompressible fluids.
8. For each of the four coordinate systems shown in Figure 6.13, what is the vector that expresses the acceleration due to gravity?
9. In Figure 6.13, the following equality is given:

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{xyz} = \begin{pmatrix} v_{\bar{x}} \\ 0 \\ v_{\bar{z}} \end{pmatrix}_{\bar{x}\bar{y}\bar{z}}$$

   Explain how both of these ways to express $\underline{v}$ are correct. Note that $v_x \neq v_{\bar{x}}$, $v_z \neq v_{\bar{z}}$.
10. Show with matrix operations on Cartesian coordinates that $\nabla \cdot p\underline{\underline{I}} = \nabla p$. Use Table B.2 in Appendix B to obtain the Cartesian coordinates of this Gibbs expression.
11. Using matrices and the definition of the gradient of a vector (Appendix B), show that the following two expressions are equivalent:

$$\nabla \cdot (\rho \underline{v}) = \underline{v} \cdot \nabla \rho + \rho(\nabla \cdot \underline{v})$$

12. Using matrices and the definition of the gradient of a vector (Appendix B), show that the following two expressions are equivalent:

$$\frac{\partial(\rho \underline{v})}{\partial t} + \nabla \cdot (\rho \underline{v}\, \underline{v}) = \rho \frac{\partial \underline{v}}{\partial t} + \underline{v} \frac{\partial \rho}{\partial t} + \rho\,(\underline{v} \cdot \nabla \underline{v}) + \underline{v} \nabla \cdot (\rho \underline{v})$$

13. Using matrices and the definition of the gradient of a vector (Appendix B), show that the following two expressions are equivalent (viscosity is constant):

$$\nabla \cdot \left(\mu \left(\nabla \underline{v} + (\nabla \underline{v})^T\right)\right) = \mu \nabla^2 \underline{v} + \mu \nabla (\nabla \cdot \underline{v})$$

14. Using matrices and the definition of the gradient of a vector (Appendix B), show that the following two expressions are equivalent:

$$\frac{\partial\left(\rho\hat{E}\right)}{\partial t}+\nabla\cdot\left(\underline{v}\rho\hat{E}\right)=\rho\frac{\partial\hat{E}}{\partial t}+\hat{E}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\underline{v})\right]+\rho\left(\underline{v}\cdot\nabla\hat{E}\right)$$

15. Using matrices and the definition of the gradient of a vector (Appendix B), show for a Newtonian fluid that $\underline{\underline{\tilde{\tau}}}^T:\nabla\underline{v}$ is always positive. This term appears in the derivation of the microscopic energy balance [108].
16. Compare and contrast the form of the momentum balance in the Navier-Stokes equation and the form of the momentum balance given by Newton's second law of motion, $\sum\underline{f}=m\underline{a}$.
17. What is the difference between the equation of motion with the extra-stress tensor $\underline{\underline{\tilde{\tau}}}$ included (i.e., the Cauchy momentum equation, Equation 6.65) and the equation of motion with viscosity $\mu$ present (i.e., the Navier-Stokes equation, Equation 6.71)? Can both equations be used for Newtonian fluids?
18. In the solution method for microscopic-momentum-balance problems outlined in Section 6.2, how would the solution steps change if the fluid under consideration were non-Newtonian rather than Newtonian?
19. A fluid flows down an inclined plane. The magnitude of the total force on the plane is $100N$. If a fluid of the same density but 10 times higher viscosity flows down the incline, what is the magnitude of the total force on the plane? Explain your answer.
20. If the velocity vector $\underline{v}$ in m/s and pressure in $Pa$ for water flow in a pipe (radius 0.010 m, length 2.00 m) are given by the following expressions, what is the vector $\underline{\mathcal{F}}$ that indicates the magnitude and direction of the force on the walls of the pipe?

$$p[Pa]=-240z$$

$$\underline{v}[m/s]=\begin{pmatrix}0\\0\\6.0\left(1-\left(\frac{r}{0.010}\right)^2\right)\end{pmatrix}_{r\theta z}$$

where $r$ and $z$ are expressed in m.

21. If the velocity vector $\underline{v}$ in m/s for flow through a pipe of radius 0.012 m is given by the following expression, what is the volumetric flow rate $Q$ of fluid through the pipe?

$$\underline{v}[m/s]=\begin{pmatrix}0\\0\\12.0\left(1-\left(\frac{r}{0.012}\right)^2\right)\end{pmatrix}_{r\theta z}$$

where $r$ is expressed in m.

22. Fluid is trapped between two concentric cylinders and the inner cylinder (with radius $=\kappa R$) is turning, producing the velocity field $\underline{v}$ given here. What is the torque on the inner cylinder? What is the torque on the

outer cylinder (with radius $= R$)? Assume that the pressure is constant throughout.

$$\underline{v} = \begin{pmatrix} 0 \\ \left(\frac{\kappa^2 \Omega R}{\kappa^2 - 1}\right)\left(\frac{r}{R} - \frac{R}{r}\right) \\ 0 \end{pmatrix}_{r\theta z}$$

23. For a pressure-driven flow in a slit, the total stress tensor $\underline{\underline{\tilde{\Pi}}}$ is given here, where $P = Cx$ is pressure, $\mu$ is viscosity, $B$ is the gap half-height, and $A$ is the velocity at the centerline. The fluid is an incompressible Newtonian fluid. What is the $x$-component of the force due to fluid on the bottom plate?

$$v_x(z) = A\left(1 - \frac{z^2}{B^2}\right)$$

$$\underline{\underline{\tilde{\Pi}}} = \begin{pmatrix} -P & 0 & -\frac{2\mu A z}{B^2} \\ 0 & -P & 0 \\ -\frac{2\mu A z}{B^2} & 0 & -P \end{pmatrix}_{xyz}$$

24. For water in a flow with the velocity vector given here, what is the force in the fluid on a square surface with unit normal $\hat{n} = \hat{e}_z$ extending from $x = 0$, $y = 0$ to $x = 1$, $y = 1$. All distances are in meters; assume the pressure is the same everywhere.

$$\underline{v}[m/s] = \begin{pmatrix} -0.04x \\ -0.04y \\ 0.08z \end{pmatrix}_{xyz}$$

25. What is the torque on a rod turning in an infinite bath of fluid? The radius of the rod is $R$, the length is $L$, and the rod turns at angular velocity $\Omega$ in a fluid of viscosity $\mu$. You may leave your answer in terms of the unknown velocity distribution.
26. The velocity field for squeeze flow between parallel plates is given here (This was obtained with a quasi-steady-state solution [12]), where $h$ is the instantaneous gap height. What is the instantaneous flow rate through the circular strip of surface of height $2h$ at $r = R/2$? The area of this surface is $2\pi(R/2)2h$.

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} -\frac{3}{4}\frac{Vr}{h}\left(\frac{z^2}{h^2} - 1\right) \\ 0 \\ \frac{3}{2}V\left(\frac{z^3}{3h^3} - \frac{z}{h}\right) \end{pmatrix}_{r\theta z}$$

27. A fluid in a circular tank is in solid-body rotation on a turntable. The velocity field is given here. What is the fluid force on the wall due to the rotation? Explain your results.

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} 0 \\ r\Omega \\ 0 \end{pmatrix}_{r\theta z}$$

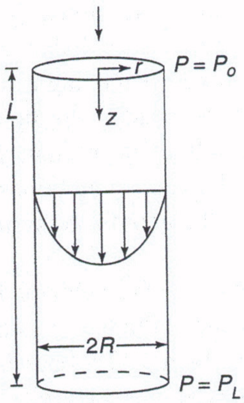


Figure 6.20 Pressure-driven flow in a tube (see Problem 30).

28. The $z$-direction velocity shown here results from the effect of natural convection (i.e., hot air rises) between two vertical parallel plates (the plates are long and wide). At what position in the flow does the velocity reach a maximum? In the equation, $v_z$ is the velocity and $y$ is the coordinate direction in a Cartesian coordinate system; all other quantities are constants related to the flow.

$$v_z(y) = \frac{\bar{\rho}\bar{\beta}(T_2 - T_1)b^2}{12\mu}\left[\left(\frac{y}{b}\right)^3 - \left(\frac{y}{b}\right)\right]$$

29. For the combined pressure-driven and wall-drag flow discussed in Example 6.11, sketch the coordinate system that was used to find the velocity solution provided in the example. What are the boundary conditions that were used?
30. What are the velocity boundary conditions in the flow shown in Figure 6.20? Express your answer in the coordinate system shown.
31. The problem of a thin film falling down an incline is discussed in this chapter. In Figure 5.41, a version of this problem is illustrated and a Cartesian coordinate system is proposed. Write the velocity vector in this coordinate system. How does the continuity equation simplify? How does the Navier-Stokes equation simplify? Comment on the chosen coordinate system.
32. For the velocity described in Figure 5.31 (i.e., the planar-jet flow), apply the microscopic mass balance (i.e., the continuity equation). What is the relationship between the velocity gradients that the mass balance requires?
33. Two different fluids with different densities and viscosities are layered between two long, parallel plates. The thickness of the bottom and more dense fluid layer is $h_1$; the thickness of the top and less dense fluid layer is $h_2$. The top plate is made to move parallel to the bottom plate at a low velocity $V$. The bottom plate is stationary. The flow is steady and both fluids are incompressible. The flow problem is solved in a Cartesian coordinate system with flow in the $x$-direction, and $y$ is the direction perpendicular to the plates, with $y = 0$ at the surface of the bottom plate. What are the boundary conditions for this flow? Give your answer in mathematical form in the coordinate system described.

34. An incompressible fluid with density $\rho$ and viscosity $\mu$ is placed in the region between two coaxial cylinders. The outer cylinder (radius $= R$) is stationary and the inner cylinder (radius $= \kappa R$) is moving counterclockwise at angular velocity $\Omega$ in rad/s. The flow is steady. The flow is solved in a cylindrical coordinate system with $z = 0$ at the bottom surface of the apparatus; $\hat{e}_z$ points upward. What are the boundary conditions for this flow? Give the answer in mathematical form in the coordinate system described. Check the units of your expressions.
35. In Example 6.2, we discuss the solution for the velocity field for flow down an inclined plane. The upper boundary of this flow is a free surface, meaning that there is no solid surface there. The boundary condition at the free surface where two fluids meet is that the velocity and stress should be continuous across the boundary. Consider the free surface in the flow-down-an-incline problem as the meeting point of two fluids—air and water—with the viscosity of air being much lower than the viscosity of water. Using the stress-matching boundary condition, justify the boundary condition used for the free surface in Example 6.2.
36. Sketch the flow domain for upward flow in a circular pipe inclined by a 30-degree angle to the horizontal. Pipe flow usually is analyzed in a cylindrical coordinate system. In terms of the cylindrical coordinate system for this problem, what is the gravity vector? Hint: choose gravity to be in the x-z plane. What are the implications of this complicated expression? How would this complication affect the solution of the Navier-Stokes equations for this problem?
37. Figure 6.12c depicts the steady flow of a drop of Newtonian fluid "rolling" down an inclined plane. Because of the complex geometry of the flow domain, this flow is best analyzed numerically, and the details of that calculation are beyond the scope of this text [70]. Although numerical methods are needed to solve the differential equations, we can arrive at the correct equations to solve by following the methods in this text. What is the differential equation that governs this flow and what are the appropriate boundary conditions?
38. *Flow Problem: Drag flow of a Newtonian fluid in a slit.* Calculate the velocity profile and flow rate for drag flow of an incompressible Newtonian liquid between two infinitely wide parallel plates separated by a gap of $H$. The pressure in the gap is uniform in the flow direction. The lower plate does not move, but the upper plate is pulled to the right at a speed $V$. The flow is steady and well developed.
39. *Flow Problem: Pressure-driven flow in an uphill slit.* An incompressible Newtonian fluid is made to flow between two long, wide parallel plates by a constant driving pressure gradient. The pressure at an upstream point is $P_0$ and a distance L downstream the pressure is $P_L$. The plates tilt upwards, making an angle $\psi$ with the horizontal; do not neglect gravity. Calculate the steady state velocity profile, the flow rate, and the force on the walls. The gap between the plates is B.
40. *Flow Problem: Combined forward pressure and drag.* An incompressible Newtonian fluid is made to flow between two long, wide, horizontal parallel plates by the combined effect of a constant driving pressure gradient and

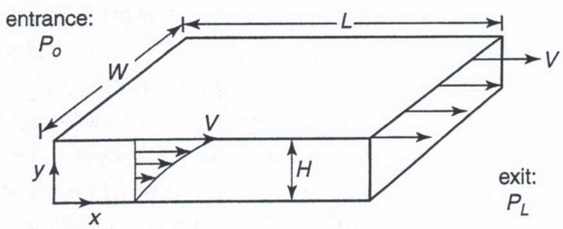


Figure 6.21 Schematic for Problem 41.

the motion of the wall. The gap between the plates is B, and the top plate is pulled to the right at a speed $V$ while the lower plate remains stationary. The pressure at an upstream point is $P_0$ and a distance $L$ downstream the pressure is $P_L$. Calculate the steady state velocity field and the flow rate.

41. *Flow Problem: Combined backward pressure and forward drag of a Newtonian fluid in a slit.* Calculate the velocity profile for the flow shown in Figure 6.21. The flow is steady flow of an incompressible Newtonian fluid between two wide plates. The flow is driven forward by the motion of the top plate (i.e., the top plate moves in the $x$-direction at speed $V$) and the flow is opposed by the pressure, which is slightly higher at the exit, $P_L$, than at the entrance, $P_0$, $P_L > P_0$. Neglect the effect of gravity. Use the coordinate system given in Figure 6.21.
42. *Flow Problem: Combined pressure-driven/drag flow of a Newtonian fluid in a slit that is tilted upward.* Calculate the velocity profile and flow rate for pressure-driven flow of an incompressible Newtonian liquid between two infinitely wide parallel plates separated by a gap of $H$. The slit is inclined to the horizontal by an angle $\alpha$. The top plate moves forward at velocity $V$. The pressure at an upstream point is $P_0$; at a point a distance $L$ downstream, the pressure is $P_L$. Assume that the flow between the plates is well developed and at steady state. The axial pressure gradient is constant.
43. *Flow Problem: Axial annular drag, wire coating.* An incompressible Newtonian fluid fills the annular gap between a cylindrical wire of radius $\kappa R$ and an outer shell of inner radius $R$ (Figure 6.22). The wire is pulled to the

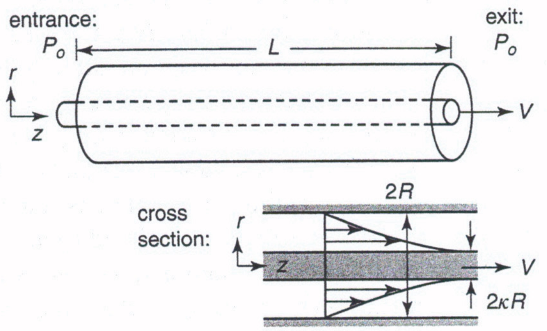


Figure 6.22 In wire-coating, a wire is drawn through a bath. This axial drag flow is addressed in Problem 43.

right at a speed V. There is no pressure variation throughout the apparatus. Calculate the steady state velocity profile, the flow rate, and the force on the wire. This geometry occurs in wire-coating. Answer: $v_z/V = \ln(r/R)/\ln\kappa$.

44. *Flow Problem: Upward, pressure-driven flow of a Newtonian fluid in a pipe.* Calculate the velocity profile and flow rate for pressure-driven flow of an incompressible Newtonian liquid in a vertical pipe of radius $R$. The pressure at the bottom entrance to the tube is $P_0$; at a point a distance $L$ upward, the pressure is $P_L$. Assume that the flow is well developed and at steady state. Do not neglect gravity.
45. *Flow Problem: Two-layer drag flow between parallel plates.* Two different fluids with different densities and different viscosities are layered between two long, parallel plates. The thickness of the bottom and more dense fluid is $h_1$; the thickness of the top and less dense fluid is $h_2$. The top plate is made to move parallel to the bottom plate at a velocity $V$. The bottom plate is stationary. The flow is steady and both fluids are incompressible. Solve for the velocity profile in a Cartesian coordinate system with flow in the $x$-direction; $y$ is the direction perpendicular to the plates, with $y = 0$ at the surface of the bottom plate.
46. *Flow Problem: Two-layer drag, pressure-driven flow between parallel plates.* Two different fluids with different densities $(\rho_1, \rho_2)$ and viscosities $(\mu_1, \mu_2)$ are layered between two long, parallel plates. The thickness of the bottom and more dense fluid layer is $h_1$; the thickness of the top and less dense fluid layer is $h_2$. Both plates are stationary and a flow is produced by the imposition of a small, constant pressure gradient such that the interface between the two fluids remains flat and parallel to the walls. The flow is steady and both fluids are incompressible. Solve for the velocity profile in a Cartesian coordinate system with flow in the $x$-direction; $y$ is the direction perpendicular to the plates, with $y = 0$ at the surface of the bottom plate (copious algebra!).
47. *Flow Problem: Two-layer flow down an incline.* Two different fluids with different densities $(\rho_1, \rho_2)$ and viscosities $(\mu_1, \mu_2)$ are layered on a long plate tilted at an angle $\beta$ to the horizontal. The thickness of the bottom and more dense fluid layer is $h_1$; the thickness of the top and less dense fluid layer is $h_2$. The bottom plate is stationary and a flow is produced by gravity. The flow is steady and both fluids are incompressible. Solve for the velocity profile in a Cartesian coordinate system with flow in the $x$-direction; $y$ is the direction perpendicular to the plates, with $y = 0$ at the surface of the bottom plate.
48. *Flow Problem: Drag flow with viscosity varying.* Calculate the velocity profile and flow rate for drag flow of an incompressible Newtonian liquid between two infinitely wide parallel plates separated by a gap of $H$. The viscosity of the fluid varies linearly with position in the gap as $\mu = ay + b$. The pressure in the gap is uniform in the flow direction. The upper plate is driven such that the velocity is $V$ and the lower plate is stationary. Assume that the flow between the plates is well developed and at steady state. Solve for the velocity profile in a Cartesian coordinate system with flow in the $x$-direction; $y$ is the direction perpendicular to the plates, with $y = 0$ at the surface of the bottom plate.

49. *Flow Problem: Axial annular drag with pressure drop, wire coating.* Repeat Problem 43 with an imposed pressure gradient $= (-\Delta p/L)$ in the flow direction. Calculate the velocity field only.
50. *Flow Problem: Drag flow in a slit, power-law non-Newtonian fluid.* Repeat Problem 38 with a power-law, generalized Newtonian fluid with parameters $m$ and $n$. Calculate the velocity and stress fields.
51. *Flow Problem: Pressure-driven flow in a slit tilted upward, power-law non-Newtonian fluid.* Repeat Problem 39 with a power-law, generalized Newtonian fluid with parameters $m$ and $n$. Calculate the velocity field only.
52. *Flow Problem: Upward pressure-driven flow in a tube, power-law non-Newtonian fluid.* Repeat Problem 44 with a power-law, generalized Newtonian fluid with parameters $m$ and $n$.