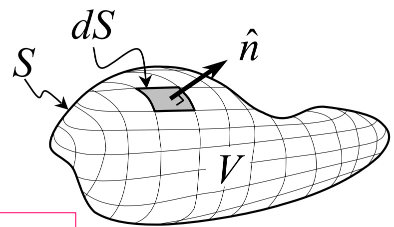
General Energy Transport Equation
(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, V, enclosed by a surface, S.



Gibbs notation:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

see handout for component notation

General Energy Transport Equation
(microscopic energy balance)

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

- $\frac{\partial T}{\partial t}$: rate of change
- $\underline{v} \cdot \nabla T$: convection
- $k \nabla^2 T$: conduction (all directions)
- S: source (energy generated per unit volume per time)

velocity must satisfy equation of motion, equation of continuity

see handout for component notation

Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

CM310 Fall 1999 Faith Morrison

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = \frac{k}{\rho\hat{C}_p}\nabla^2 T + \frac{S}{\rho\hat{C}_p}$$

Note: this handout is on the web

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z} = \frac{k}{\rho\hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho\hat{C}_p}$$

Cylindrical (rθz) coordinates:

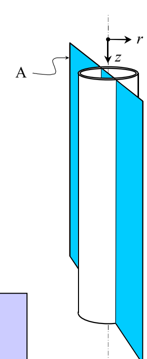
$$\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z} = \frac{k}{\rho\hat{C}_p}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho\hat{C}_p}$$

Spherical (rθϕ) coordinates:

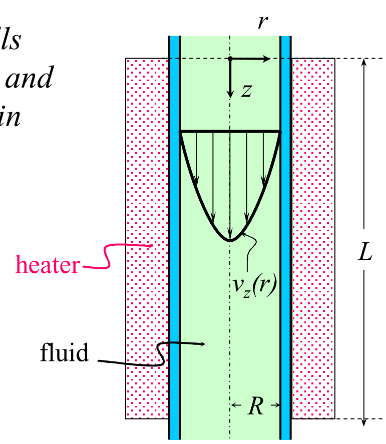
$$\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} = \frac{k}{\rho\hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2}\right.$$

** Revisit from lecture 9 ** Revisit from lecture 9 **

Example 6: **Wall heating of laminar flow**. *What is the steady state temperature profile in a flowing fluid in a tube if the walls are heated (constant flux, q_1/A) and if the fluid is a Newtonian fluid in laminar flow?*



cross-section A:



It's easy to arrive at correct PDE using micro-E-balance eqn

Microscopic Energy Balance – is the correct physics for all problems!

Tricky step:

solving for *T* field; this can be mathematically difficult

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- may not be separable from mass and momentum balances

Strategy: solve using numerical methods (e.g. ***Comsol***)

**** **Or** ****

Develop correlations on complex systems by using ***Dimensional Analysis***

There are two types of data correlations you will encounter:

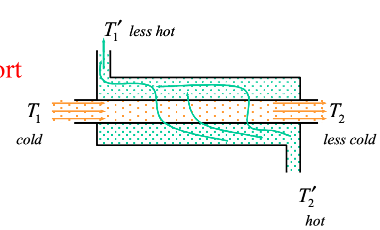
1. Standardized correlations you can look up (e.g. for *h* to use in heat exchangers)
2. In-house correlations you develop yourself (e.g. for scale-up)

Type 1: Standardized Correlations for Simple Geometries (you can look them up), e.g. heat exchanger design

$$Q = UA\Delta T$$

these are easy to sort out

but what about this? For design purposes, where do we get U?



ANSWER: Begin at the beginning: how did we define (develop) *U*?

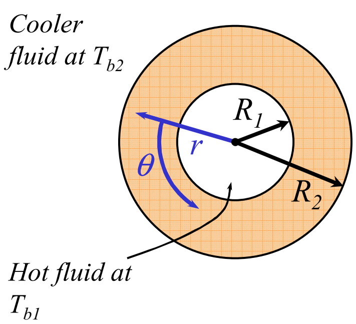
Heat exchanger design

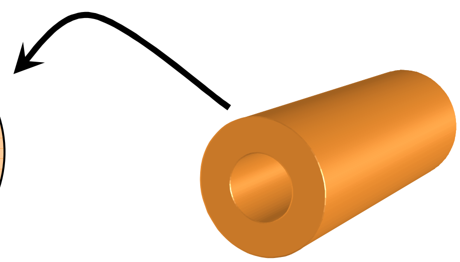
** REVIEW OF LECTURE 4 ** REVIEW OF LECTURE 4 **

Example 4: Heat flux in a cylindrical shell

Assumptions:
- long pipe
- steady state
- k = thermal conductivity of wall
- h_1, h_2 = heat transfer coefficients

What is the steady state temperature profile in a cylindrical shell (pipe) if the ***fluid on the inside*** *is at T_{b1} and the* ***fluid on the outside*** *is at T_{b2}? ($T_{b1}>T_{b2}$)*





Heat exchanger design

** REVIEW OF LECTURE 4 ** REVIEW OF LECTURE 4 **

Example 4: Heat flux in a cylindrical shell

Solution:

$$\frac{q_r}{A} = \frac{c_1}{r}$$

Not constant

$$T = -\frac{c_1}{k}\ln r + c_2$$

Boundary conditions?

Heat exchanger design

** REVIEW OF LECTURE 4 ** REVIEW OF LECTURE 4 **

Example 4: Heat flux in a cylindrical shell

$$\frac{c_1}{R_1} = h_1\left(T_{b1} - T_{w1}\right)$$

$$\frac{c_1}{R_2} = h_2\left(T_{w2} - T_{b2}\right)$$

$$T_{w1} = -\frac{c_1}{k}\ln R_1 + c_2$$

$$T_{w2} = -\frac{c_1}{k}\ln R_2 + c_2$$

4 equations

4 unknowns; c_1, T_{w1}, c_2, T_{w2}

SOLVE

Heat exchanger design

** REVIEW OF LECTURE 4 ** REVIEW OF LECTURE 4 **

Example 4: Heat flux in a cylindrical shell

Results: Radial Heat flux in an Annulus

$$T - T_{b2} = \frac{\left(T_{b1} - T_{b2}\right)\left(\ln\left(\frac{R_2}{r}\right) + \frac{k}{h_2 R_2}\right)}{\frac{k}{h_2 R_2} + \ln\left(\frac{R_2}{R_1}\right) + \frac{k}{h_1 R_1}}$$

$$\frac{q_r}{A} = \left(\frac{\left(T_{b1} - T_{b2}\right)}{\frac{1}{h_2 R_2} + \frac{1}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_1 R_1}}\right)\left(\frac{1}{r}\right)$$

Heat exchanger design

Example 4: Heat flux in a cylindrical shell

Flux:

$$\frac{q_r}{A} = \left(\frac{(T_{b1} - T_{b2})}{\frac{1}{h_2 R_2} + \frac{1}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_1 R_1}} \right) \left(\frac{1}{r} \right)$$

Total Heat flow:

$$Q = \left(\frac{q_r}{A} \right) 2\pi r L$$

$$= \frac{(T_{b1} - T_{b2}) \, 2\pi L}{\frac{1}{h_2 R_2} + \frac{1}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_1 R_1}}$$

note that total heat flow is proportional to **bulk ΔT** and (almost) **area of heat transfer**

Overall Heat Transfer Coefficient, **U**

$$Q = UA\Delta T$$

$$= UA(T_{b1} - T_{b2})$$

this equation serves as the definition of U

$$\Delta T_{driving\ force}$$

Heat exchanger design

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**** REVIEW OF LECTURE 4 ** REVIEW OF LECTURE 4 ****

overall heat xfer coeffs in pipe

$$Q = U_1 A_1 \Delta T = \left(\frac{\frac{1}{R_1}}{\frac{1}{h_2 R_2} + \frac{1}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_1 R_1}} \right)(2\pi R_1 L)(T_{b1} - T_{b2})$$

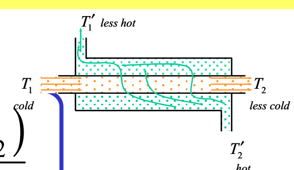
Area must be specified when U is reported

$\Delta T_{driving\ force}$

In Lecture 4 we derived a **design equation for heat exchangers** based on a constant bulk temperature difference – (driving force for heat transfer)

Heat exchanger design

**** REVIEW OF LECTURE 7 ** REVIEW OF LECTURE 7 ****



FINAL RESULT:

$$Q = \underbrace{U(2\pi RL)}\frac{(T_1'-T_1)-(T_2'-T_2)}{\ln\frac{(T_1'-T_1)}{(T_2'-T_2)}}$$

$\equiv \Delta T_{lm}$

$\Delta T_{driving\ force}$

$$Q = UA\Delta T_{lm}$$

In Lecture 7 we showed that for constant *U* (independent of temperature difference) we could expand the use of the **design equation for heat exchangers** to double-pipe heat exchangers with the appropriate new driving force ΔT_{lm}

Heat exchanger design

overall heat xfer coeffs in pipe

$$Q = U_1 A_1 \Delta T = \left(\frac{\frac{1}{R_1}}{\frac{1}{h_2 R_2} + \frac{1}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_1 R_1}} \right) (2\pi R_1 L) \Delta T_{driving\ force}$$

The design equation for heat exchangers indicates that,

U depends on geometry (R_1, R_2), materials of construction (k), and . . . h_1, h_2.

Individual "side" heat transfer coefficients

Heat exchanger design

To design a heat exchanger, we need to specify geometry and materials of construction; we also need data to know the h's

$$Q = U_1 A_1 \Delta T = \frac{\frac{1}{R_1}}{\frac{1}{h_2 R_2} + \frac{1}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_1 R_1}} (2\pi R_1 L)(\Delta T_{driving\ force})$$

We need data correlations

We will (soon) show how one develops a data correlation for h

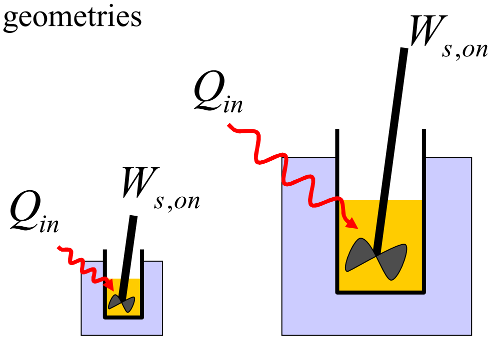
Heat exchanger design

Complex equipment design

The other type of data correlations are,

Type 2: Home-made correlations for complex geometries

- Bench-scale expts yield data correlations for Q_{in}, $W_{s,on}$
- Will match performance of full-sized process unit if scaled properly

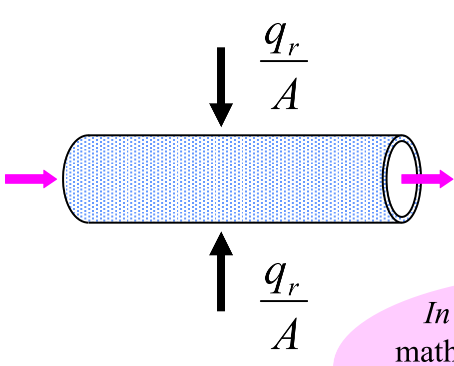


Bench scale

Process scale

Now: How do we use Dimensional Analysis to develop correlations for *h*?

Consider: Heat-transfer to fluid inside of a heated tube – forced-convection heat transfer plus radial conduction



T_1= core temperature
T_o= wall temperature
$T(r,\theta,z)$ = fluid temp distribution

In principle, with the right math/computer tools, we could calculate the complete temperature and velocity profiles.

(we already figured out what the energy equation would simplify to in Lecture 9)

Now: How do we use Dimensional Analysis to develop correlations for h?

In principle, with the right math/computer tools, we could calculate the complete temperature and velocity profiles.

How: From the microscopic mass, momentum, and energy balances, calculate velocity field, temperature field, and calculate the total amount of heat transferred. Finally, determine h from its definition.

From the complete solution, we could then calculate *h:*

$$h = \frac{q_r}{A\left(T_{bulk} - T_{wall}\right)}$$

Integrate heat flux across the total surface area; heat flux comes from Fourier's law; Fourier's law needs T field

Therefore, the equations we need for the complete solution, and the equation we use to define h contain all the physics of h; dimensional analysis on these equations will tell us what h is a function of

Now: How do we use Dimensional Analysis to develop correlations for h?

Therefore, the equations we need for the complete solution, and the equation we use to define h contain all the physics of h; dimensional analysis on these equations will tell us what h is a function of

Once we know what h is a function of, we can conduct experiments, measure h as a function of its variables, and report these data correlations in the literature for others to use.

Dimensional Analysis

principle: even in complex systems, the same equations still apply:

governing equations

- *continuity equation* (mass conservation)
- *equation of motion* (momentum conservation)
- *equation of energy* (energy conservation)

strategy: render the governing equations (and boundary conditions) dimensionless to identify the important parameters that apply in every situation.

$\Longrightarrow$ rely on experiments and data correlations

Dimensional Analysis Procedure:

1. select appropriate differential equations and boundary conditions
2. select characteristic quantities with which to scale the variables, e.g. $\underline{v}$, x, P, $T-T_o$
   - characteristic quantities must be constant
   - must be representative of the system

3. scale all variables in the governing equations; yields dimensionless equation as a function of ***dimensionless groups***
The values of the dimensionless groups determine the properties of the differential equations.

4. design scaled-down experiments to develop ***data correlations*** for the system of interest
5. use data correlations to design and evaluate systems

OR

4. perform experiments on an existing system and ***correlate*** results using dimensionless groups

Energy equation in Cartesian Coordinates:

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

Next step: choose scale factors and scale each variable.

non-dimensional variables:

time:

$$t^* \equiv \frac{tV}{D}$$

position:

$$x^* \equiv \frac{x}{D}$$

$$y^* \equiv \frac{y}{D}$$

$$z^* \equiv \frac{z}{D}$$

velocity:

$$v_x^* \equiv \frac{v_x}{V}$$

$$v_y^* \equiv \frac{v_y}{V}$$

$$v_z^* \equiv \frac{v_z}{V}$$

driving force:

$$P^* \equiv \frac{P}{\rho V^2}$$

$$g_z^* \equiv \frac{g_z}{g}$$

temperature:

$$T^* \equiv \frac{T - T_o}{(T_1 - T_o)}$$

Energy equation in Cartesian Coordinates:

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

we substitute the non-dimensional variables. e.g.

$$\frac{\partial T}{\partial t} = \frac{\partial \left[(T_1 - T_o)T^* + T_o \right]}{\partial \left[\frac{t^* D}{V} \right]} = \left[\frac{(T_1 - T_o)V}{D} \right] \frac{\partial T^*}{\partial t^*}$$

after carrying out this change of variable for each term
we get the non-dimensional energy equation

Non-dimensional Energy Equation

$$\frac{\partial T^*}{\partial t^*} + v_x^* \frac{\partial T^*}{\partial x^*} + v_y^* \frac{\partial T^*}{\partial y^*} + v_z^* \frac{\partial T^*}{\partial z^*}$$

$$= \frac{k}{\rho \hat{C}_p V D}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\partial^2 T^*}{\partial z^{*2}} \right) + \left[\frac{D}{(T_1 - T_o)\rho \hat{C}_p V} \right] S$$

$\frac{1}{Pe}$
Peclet number

S^*
dimensionless energy generation

Dimensionless Groups:

Reynolds number

$$Pe = \frac{\rho \hat{C}_p V D}{k} = \underbrace{\left(\frac{\hat{C}_p \mu}{k}\right)}_{\text{Pr}} \overbrace{\left(\frac{\rho V D}{\mu}\right)}^{\text{Re}}$$

Peclet number

Prandtl number

$$Pe = \Pr \text{Re}$$

We can conclude from the energy equation that dimensionless temperature T* is a function of:

$$\text{Pe} = \Pr \text{Re}$$

$$T^* = T^*(t^*, x^*, y^*, z^*, v_x^*, v_y^*, v_z^*, \text{Pe}, S^*)$$

We know that dimensionless velocities are a function of:

$$v_i^* = v_i^*(t^*, x^*, y^*, z^*, \text{Re}, \text{Fr})$$

$$\Rightarrow T^* = T^*(t^*, x^*, y^*, z^*, \underbrace{\text{Re}, \Pr, \text{Fr}}, S^*)$$

the equations governing temperature distributions depend on only three dimensionless groups

Now: How do we use Dimensional Analysis to develop correlations for *h*?

Therefore, the equations we need for the complete solution, and the equation we use to define *h* contain all the physics of *h*; dimensional analysis on these equations will tell us what *h* is a function of

This part is now done. From the microscopic mass, momentum, and energy balances we know that h will depend on at most Re, Pr, Fr.

Now we need to check the equation that defines *h* for more dimensionless groups