CM3110
Transport I
Part II: Heat Transfer

One-Dimensional Heat Transfer - Unsteady

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Example 1: **Unsteady Heat Conduction in a Semi-infinite solid**

A very long, very wide, very tall slab is initially at a temperature T_o. At time $t = 0$, the left face of the slab is exposed to an environment at temperature T_1. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, k, density, ρ, and heat capacity, C_p.

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This is *code* for:
"Newton's law of cooling boundary conditions:"
$$|\text{flux}| = h|T_{bulk} - T_{wall}|$$

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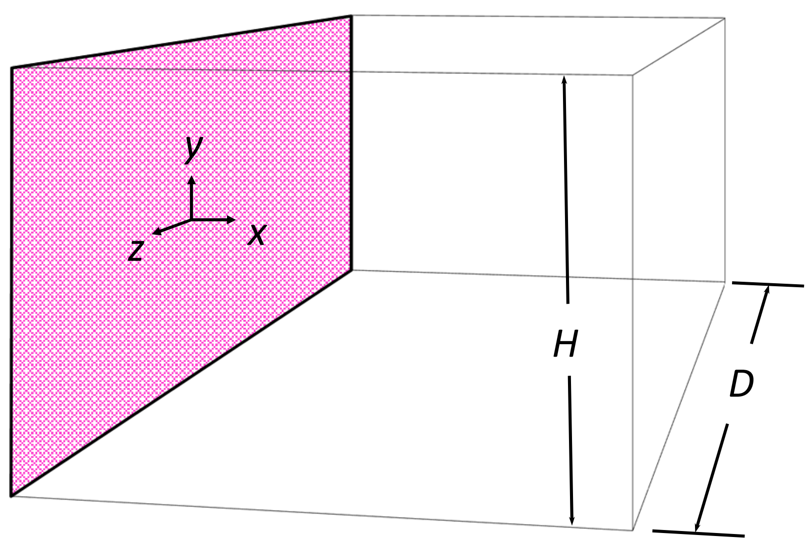
Unsteady State Heat Transfer

Example: Unsteady Heat Conduction in a Semi-infinite solid



H, D, very large

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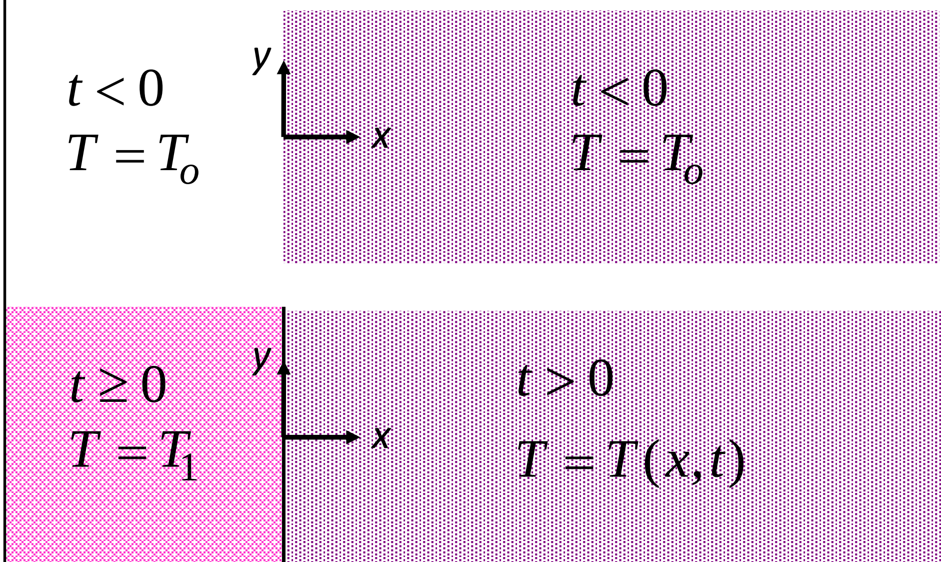
Initial Condition:



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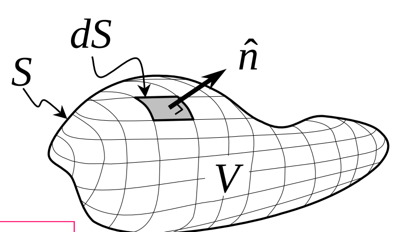
General Energy Transport Equation

(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, *V*, enclosed by a surface, *S*.



Gibbs notation:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

see handout for component notation

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General Energy Transport Equation

(microscopic energy balance)

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

- rate of change: $\frac{\partial T}{\partial t}$
- convection: $\underline{v} \cdot \nabla T$
- conduction (all directions): $k \nabla^2 T$
- source (energy generated per unit volume per time): S

velocity must satisfy equation of motion, equation of continuity

see handout for component notation

Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

CM310 Fall 1999 Faith Morrison

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

Note: this handout is on the web:
www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = \frac{k}{\rho \hat{C}_p} \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

Cylindrical (rθz) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

Spherical (rθϕ) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} = \frac{k}{\rho \hat{C}_p} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2}$$

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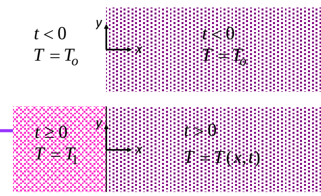

Newton's law of cooling BC's:

$$|q_x| = hA|T_{bulk} - T_{surface}|$$

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Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$$ thermal diffusivity

what are the boundary conditions? initial conditions?

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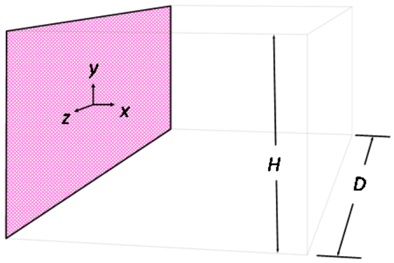
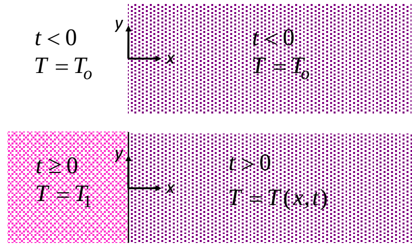
Example 7: Unsteady Heat Conduction in a Semi-infinite solid

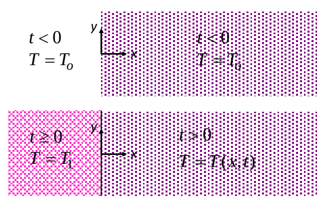


Initial Condition:



You try.

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

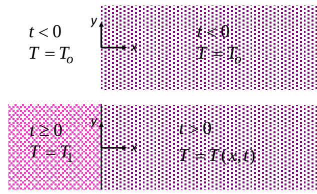
Initial condition: $t = 0, T = T_o \ \forall\ x$

Boundary conditions:

$$x = 0, \quad q_x = hA(T - T_1) \quad \forall\ t > 0$$

$$x = \infty, \quad T = T_o \quad \forall\ t$$

Unsteady State Heat Conduction in a Semi-Infinite Slab



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"for all t"

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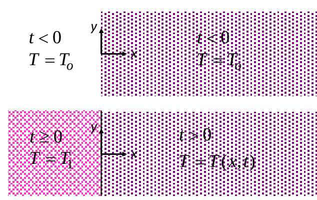
Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

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Boundary conditions:

$$x = 0, \quad q_x = hA(T - T_1) \quad \forall\ t > 0$$

$$x = \infty, \quad T = T_o \quad \forall\ t$$

The solution is obtained by combination of variables.

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

$$\zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

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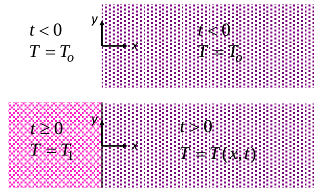
Unsteady State Heat Conduction in a Semi-Infinite Slab



Solution:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta + \beta)$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

Geankoplis 4th ed., eqn 5.3-7, page 363

complementary error function of y

$$\operatorname{erfc}(y) \equiv 1 - \operatorname{erf}(y)$$

error function of y

$$\operatorname{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-(y')^2}\, dy'$$

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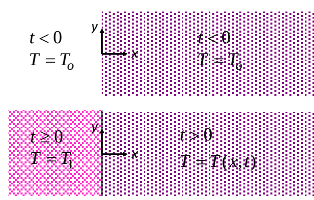
Unsteady State Heat Conduction in a Semi-Infinite Slab



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Geankoplis 4th ed., eqn 5.3-7, page 363

To make this solution easier to use, we can plot it.

complementary error function of y

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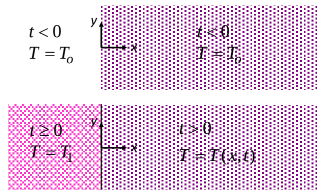
Unsteady State Heat Conduction in a Semi-Infinite Slab



This:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

Versus this: $\zeta \equiv \frac{x}{2\sqrt{\alpha t}}$

To make this solution easier to use, we can plot it.

At various values of this:

$$\beta \equiv \frac{h\sqrt{\alpha t}}{k}$$

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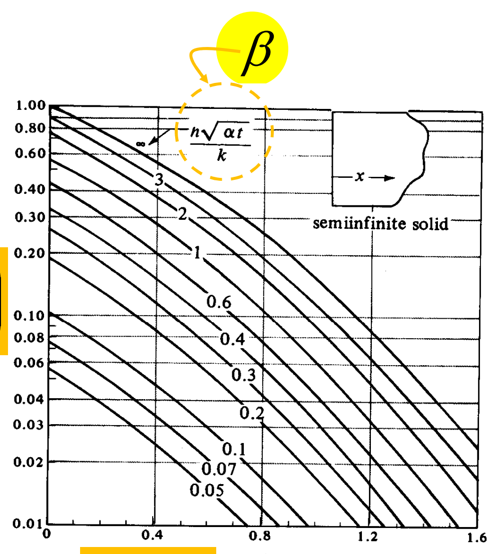
Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\left(\frac{T - T_o}{T_1 - T_o}\right)$$



$$\zeta = \frac{x}{2\sqrt{\alpha t}}$$

Geankoplis 4th ed., Figure 5.3-3, page 364

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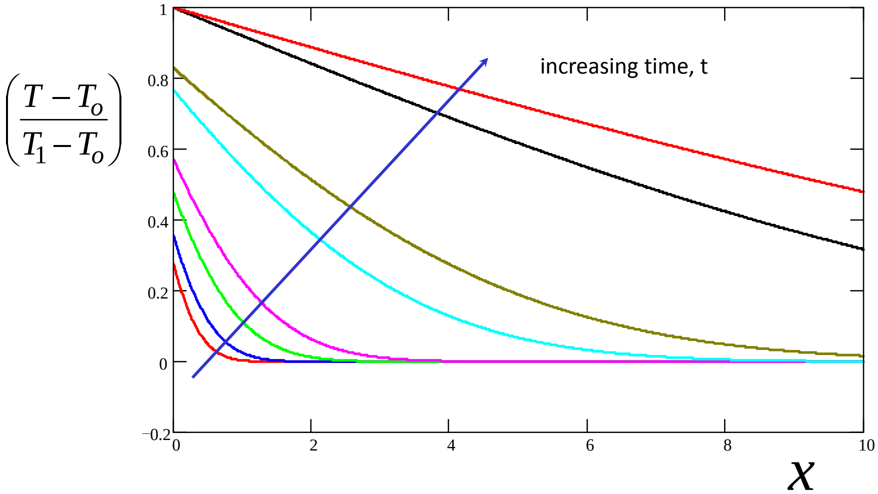
With modern tools, we can plot the solution directly (evaluated on Mathcad)

Unsteady State Heat Conduction in a Semi-Infinite Slab

$h = k = \alpha = 1$

t = 0.1, 0.2, 0.5, 1, 5, 10, 50, 100



How could we use this solution?

Example: Will my pipes freeze?

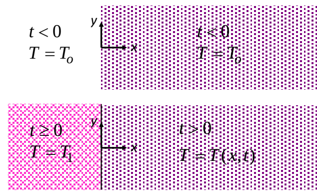
The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground? Use the following physical properties:

$$h = 2.0 \frac{BTU}{h\ ft^2\ {}^oF}$$

$$\alpha_{soil} = 0.018 \frac{ft^2}{h}$$

$$k_{soil} = 0.5 \frac{BTU}{h\ ft\ {}^oF}$$

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{T-T_0}{T_1-T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

$$\zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

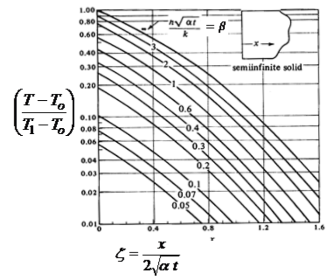
Both ζ and β depend on time

$T_0 = ?$

$T_1 = ?$

$T = ?$

$$\frac{T-T_0}{T_1-T_0} = ?$$



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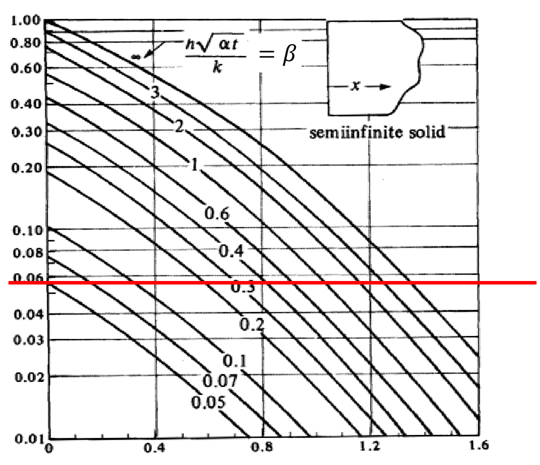
Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\left(\frac{T-T_o}{T_1-T_o}\right)$$



$$\zeta = \frac{x}{2\sqrt{\alpha\, t}}$$

(Interative solution)

Geankoplis 4th ed., Figure 5.3-3, page 364

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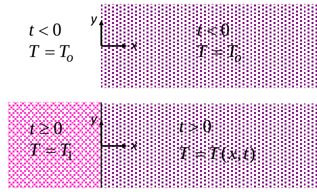
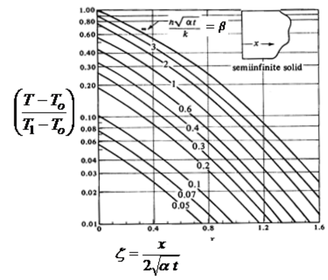
Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{T - T_0}{T_1 - T_0} = \text{erfc}\,\zeta - e^{\beta(2\zeta+\beta)}\,\text{erfc}(\zeta + \beta)$$

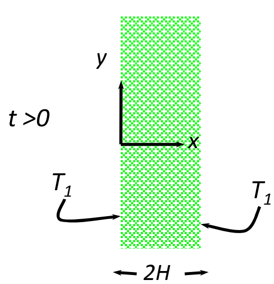


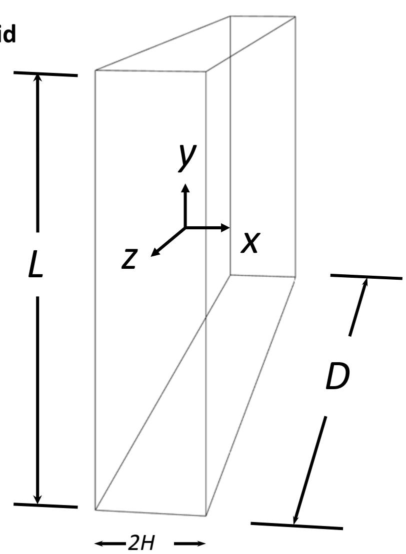
Answer:

$t = 480\ hours \approx 20\ days$

Example 8: **Unsteady Heat Conduction in a Finite-sized solid**

- The slab is tall and wide, but of thickness 2H
- Initially at T_o
- at time t = 0 the temperature of the sides is changed to T_1





Unsteady State Heat Transfer

Use same microscopic energy balance eqn as before.

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

rate of change: $\frac{\partial T}{\partial t}$

convection: $\underline{v} \cdot \nabla T$

conduction (all directions): $k \nabla^2 T$

source (energy generated per unit volume per time): S

see handout for component notation

Microscopic Energy Equation in Cartesian Coordinates

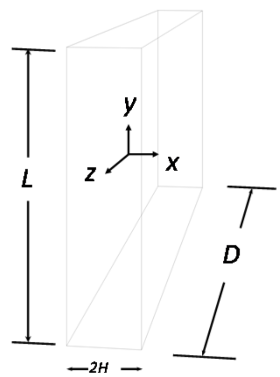
$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$$ thermal diffusivity

what are the boundary conditions? initial conditions?

Example 8: **Unsteady Heat Conduction in a Finite-sized solid**

You try.



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Unsteady State Heat Conduction in a Finite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

Initial condition: $t = 0, T = T_o \; \forall \; x$

Boundary conditions:

$$\left.\begin{array}{ll} x = 0, & T = T_1 \\ x = 2H, & T = T_1 \end{array}\right\} \quad \forall \; t > 0$$

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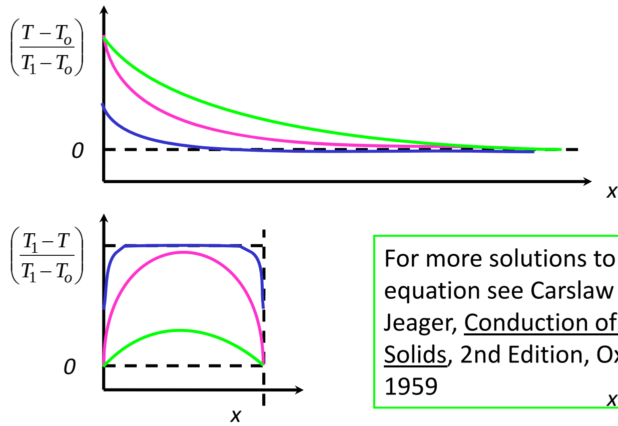
Q: *How can two completely different situations give the same governing equation?*
A: The boundary conditions make all the difference



For more solutions to this equation see Carslaw and Jeager, Conduction of Heat in Solids, 2nd Edition, Oxford, 1959

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Unsteady State Heat Conduction in a Finite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

Initial condition: $t = 0, T = T_o \ \forall \ x$

Boundary conditions:

$$\left.\begin{array}{ll} x = 0, & T = T_1 \\ x = 2H, & T = T_1 \end{array}\right\} \ \forall \ t > 0$$

The solution is obtained by separation of variables.

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Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

Let $Y \equiv \left(\frac{T_1 - T}{T_1 - T_o} \right)$

$$\frac{\partial Y}{\partial t} = \alpha \left(\frac{\partial^2 Y}{\partial x^2} \right)$$

Guess: $Y = X(x)\Theta(t)$

Initial condition:

$$t = 0, T = T_o \ \forall\ x \Rightarrow Y = 1$$

Boundary conditions:

$$\left.\begin{array}{ll} x = 0, & T = T_1 \Rightarrow Y = 0 \\ x = 2H, & T = T_1 \Rightarrow Y = 0 \end{array}\right\} \forall\ t > 0$$

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Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$Y = X(x)\Theta(t)$

$$\frac{\partial Y}{\partial t} = \alpha \left(\frac{\partial^2 Y}{\partial x^2} \right)$$

$$\frac{\partial Y}{\partial t} = \frac{\partial}{\partial t}\big(X(x)\Theta(t)\big) = X(x)\frac{d\Theta(t)}{dt}$$

$$\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x}\big(X(x)\Theta(t)\big) = \frac{dX(x)}{dx}\Theta(t)$$

$$\frac{\partial^2 Y}{\partial x^2} = \frac{d^2 X(x)}{dx^2}\Theta(t)$$

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Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$$\frac{\partial Y}{\partial t} = \alpha \left(\frac{\partial^2 Y}{\partial x^2} \right)$$

Substituting:

$$X(x)\frac{d\Theta(t)}{dt} = \alpha \frac{d^2 X(x)}{dx^2}\Theta(t)$$

$$\frac{1}{\Theta(t)}\frac{d\Theta(t)}{dt} = \alpha \frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} \Rightarrow = \lambda$$

constant

function of time (t) only

function of position (x) only

The function of two variables is separable into two functions of one variable.

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Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

Separates into two ordinary differential equations:

$$\frac{1}{\Theta(t)}\frac{d\Theta(t)}{dt} = \lambda$$

$$\alpha \frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = \lambda$$

The function of two variables is separable into two functions of one variable.

Solve.

Apply BCs.

Apply ICs.

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Temperature Profile for Unsteady State Heat Conduction in a Finite Slab

$$\left(\frac{T_1 - T}{T_1 - T_o}\right) = \frac{4}{\pi}\left\{ e^{\frac{-\pi^2 \alpha t}{4H^2}} \sin\frac{\pi x}{2H} + \frac{1}{3} e^{\frac{-3^2\pi^2\alpha t}{4H^2}} \sin\frac{3\pi x}{2H} + \frac{1}{5} e^{\frac{-5^2\pi^2\alpha t}{4H^2}} \sin\frac{5\pi x}{2H} + \cdots \right\}$$

Geankoplis 4th ed., eqn 5.3-6, p363

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Microscopic Energy Balance – is the correct physics for many problems!

Tricky step:

solving for *T* field; this can be mathematically difficult

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- may not be separable from mass and momentum balances

Strategy:
- Look up solution in literature
- solve using numerical methods (e.g. ***Comsol***)

****** Or ******

- Develop correlations on complex systems by using ***Dimensional Analysis***

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Fluid Mechanics: What did we do?

- Turbulent tube flow
- Noncircular conduits
- Drag on obstacles

1. Find a simple problem that allows us to identify the physics
2. Nondimensionalize
3. Explore that problem
4. Take data and correlate
5. Solve real problems

Solve. Real. Problems.

Powerful.

Works on heat transfer too.

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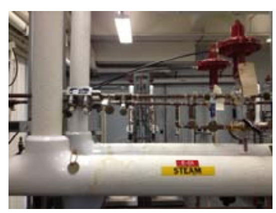
CM3110
Transport I
Part II: Heat Transfer

MichiganTech



More Complex Heat Transfer – Dimensional Analysis

Professor Faith Morrison

Department of Chemical Engineering
Michigan Technological University

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